# Moment-Distribution Method

FRAMES

Structural Analysis
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Theory of Structures-II

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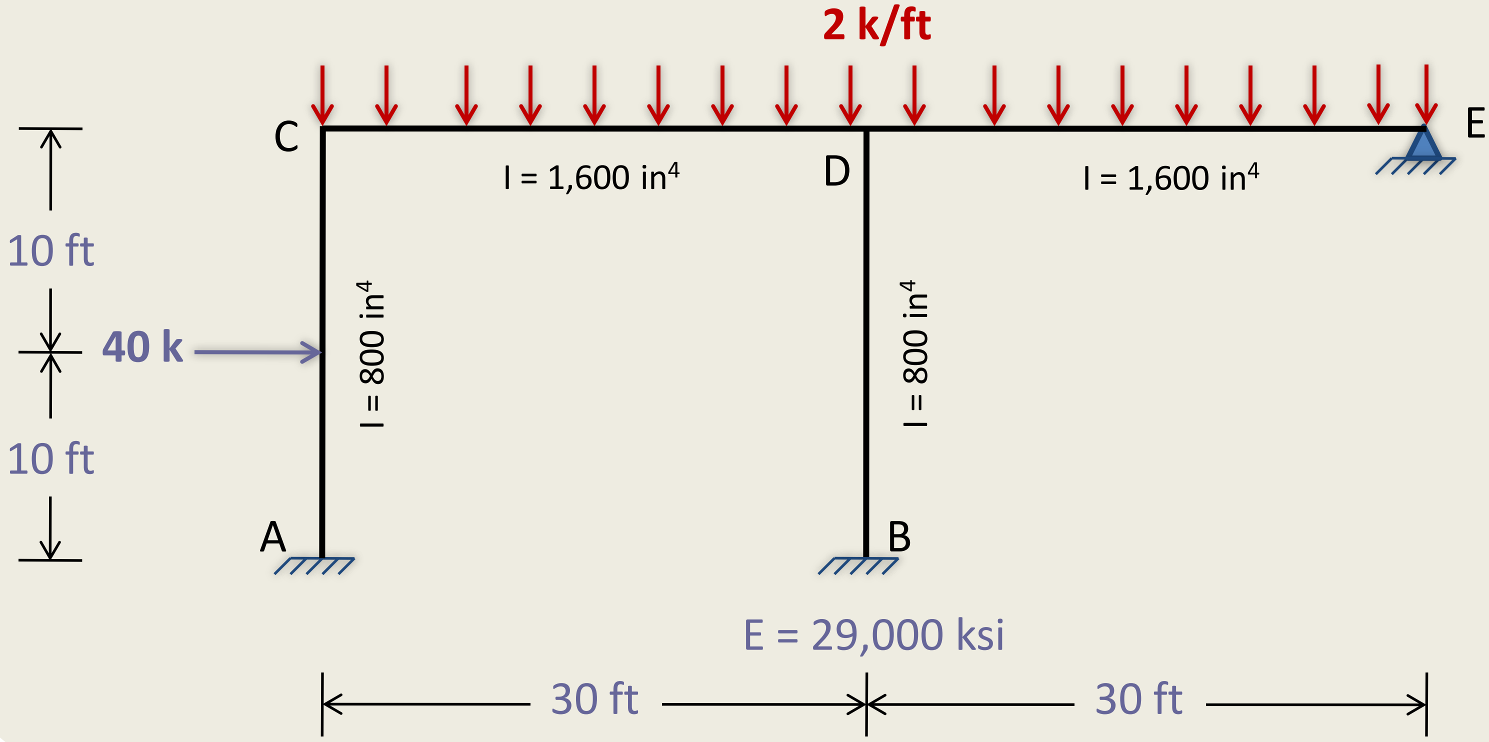
# Analysis of Frames Without Sidesway

- The procedure for the analysis of frames without sidesway is similar to that for the analysis of continuous beam.

- Unlike the continuous beams, more than two members may be connected to a joint of a frame.

# Example 1

- Determine the member end moments and reactions for the frame shown by the moment-distribution method.



# Solution

## 1.Distribution Factors

- Distribution Factors at Joint C,

$$DF_{CA}=\frac{\left(\frac{800}{20}\right)}{\left(\frac{800}{20}\right)+\left(\frac{1600}{30}\right)}=0.429$$

$$DF_{CD}=\frac{\left(\frac{1600}{30}\right)}{\left(\frac{800}{20}\right)+\left(\frac{1600}{30}\right)}=0.571$$

$$DF_{CA}+DF_{CD}=0.429+0.571=1 \qquad \text{Checks}$$

- Distribution Factors at Joint D,

$$DF_{DB} = \frac{\left(\frac{800}{20}\right)}{\left(\frac{800}{20}\right) + \left(\frac{1600}{30}\right) + \left(\frac{3}{4}\right)\left(\frac{1600}{30}\right)} = 0.3$$

$$DF_{DC} = \frac{\left(\frac{1600}{30}\right)}{\left(\frac{800}{20}\right) + \left(\frac{1600}{30}\right) + \left(\frac{3}{4}\right)\left(\frac{1600}{30}\right)} = 0.4$$

$$DF_{DE} = \frac{\left(\frac{3}{4}\right)\left(\frac{1600}{30}\right)}{\left(\frac{800}{20}\right) + \left(\frac{1600}{30}\right) + \left(\frac{3}{4}\right)\left(\frac{1600}{30}\right)} = 0.3$$

$$DF_{DB} + DF_{DE} + DF_{DC} = 2(0.3) + 0.4 = 1$$

Checks

- Distribution Factors at Joint E,

$$DF_{ED} = 1$$

## 2.Fixed-End Moments (FEMs)

$$FEM_{AC} = +100\ \mathrm{k\text{-}ft}$$

$$FEM_{CA} = -100\ \mathrm{k\text{-}ft}$$

$$FEM_{BD} = FEM_{DB} = 0$$

$$FEM_{CD} = FEM_{DE} = +150\ \mathrm{k\text{-}ft}$$

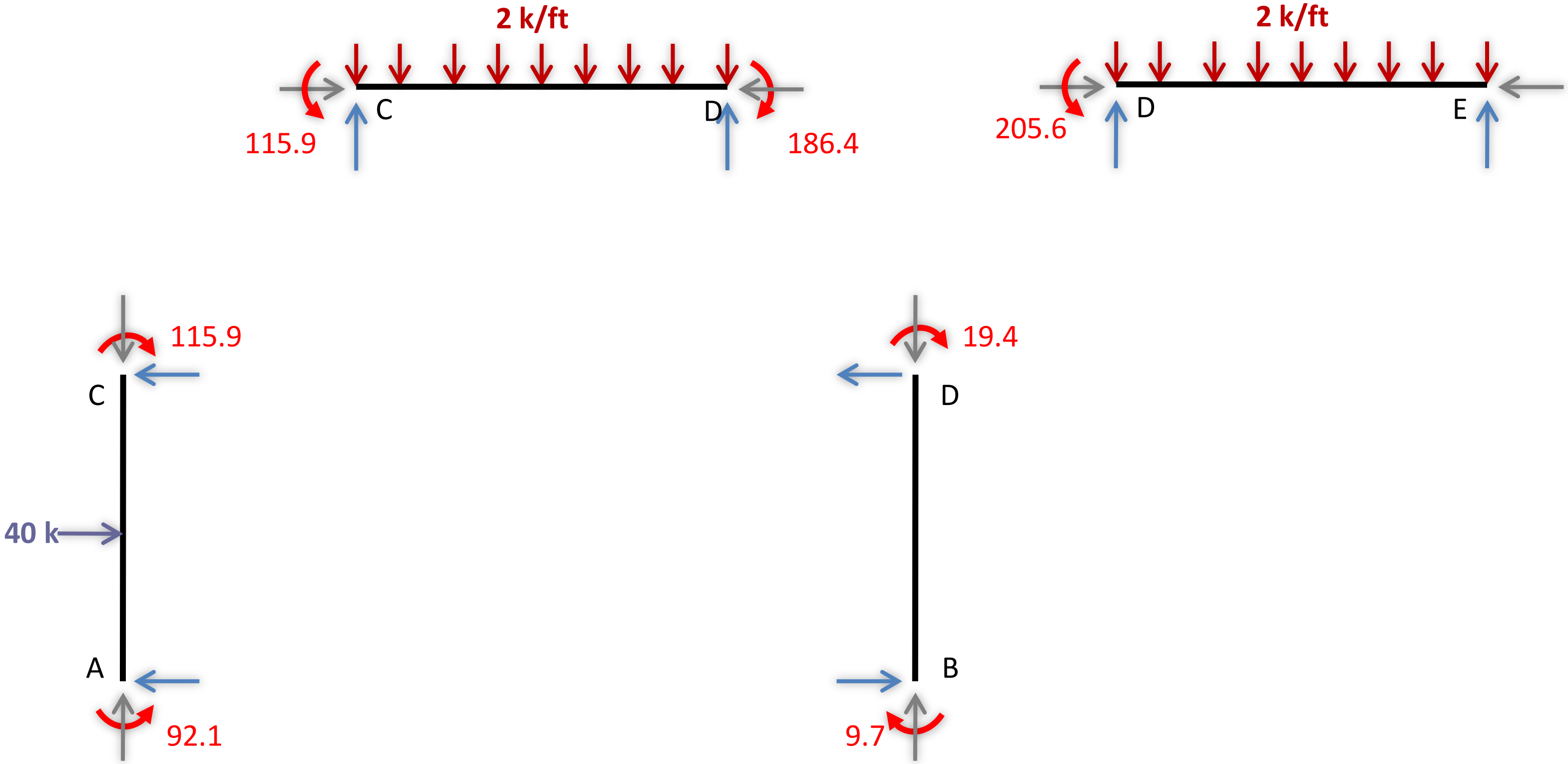
$$FEM_{DC} = FEM_{ED} = -150\ \mathrm{k\text{-}ft}$$

## 3.Moment Distribution

## 4.Final Moments

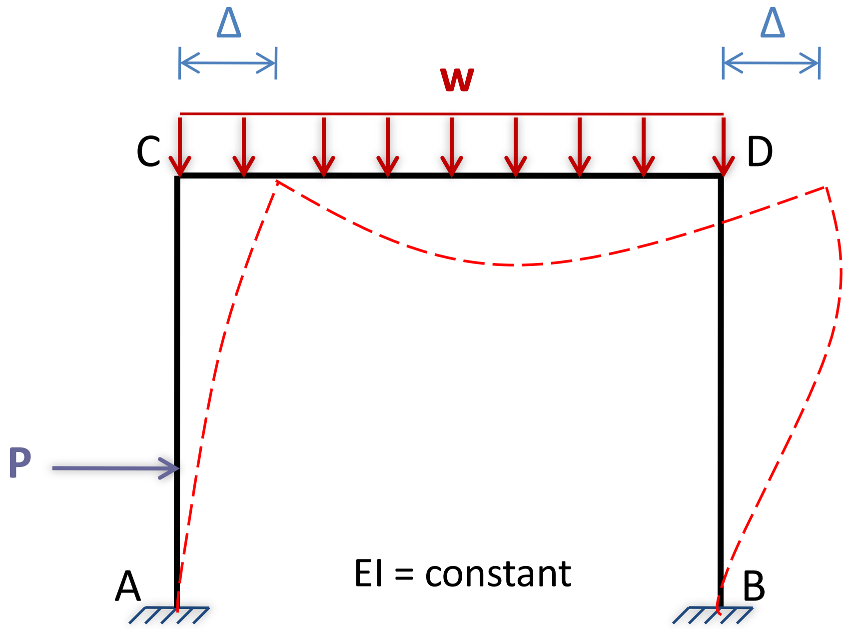
Carryover

| Member Ends | AC | CA | CD | DC | DB | DE | ED | BD |
|---|---|---|---|---|---|---|---|---|
| Distribution Factors | | 0.429 | 0.571 | 0.4 | 0.3 | 0.3 | 1 | |
| 1.Fixed-end Moments | +100 | -100 | +150 | -150 | | +150 | -150 | |
| 2.Balance Joints | | -21.4 | -28.6 | | | | +150 | |
| 3.Carryover | -10.7 | | | -14.3 | | +75 | | |
| 4.Balance Joints | | | | -24.3 | -18.2 | -18.2 | | |
| 5.Carryover | | | -12.2 | | | | | -9.1 |
| 6.Balance Joints | | +5.2 | +7 | | | | | |
| 7.Carryover | +2.6 | | | +3.5 | | | | |
| 8.Balance Joints | | | | -1.4 | -1.1 | -1.1 | | |
| 9.Carryover | | | -0.7 | | | | | -0.6 |
| 10.Balance Joints | | +0.3 | +0.4 | | | | | |
| 11. Carryover | +0.2 | | | +0.2 | | | | |
| 12.Balance Joints | | | | -0.1 | -0.1 | -0.1 | | |
| **13.Final Moments** | **+92.1** | **-115.9** | **+115.9** | **-186.4** | **-19.4** | **+205.6** | **0** | **-9.7** |


2 k/ft
C
D
115.9
186.4
2 k/ft
D
E
205.6
115.9
C
40 k
A
92.1
19.4
D
B
9.7

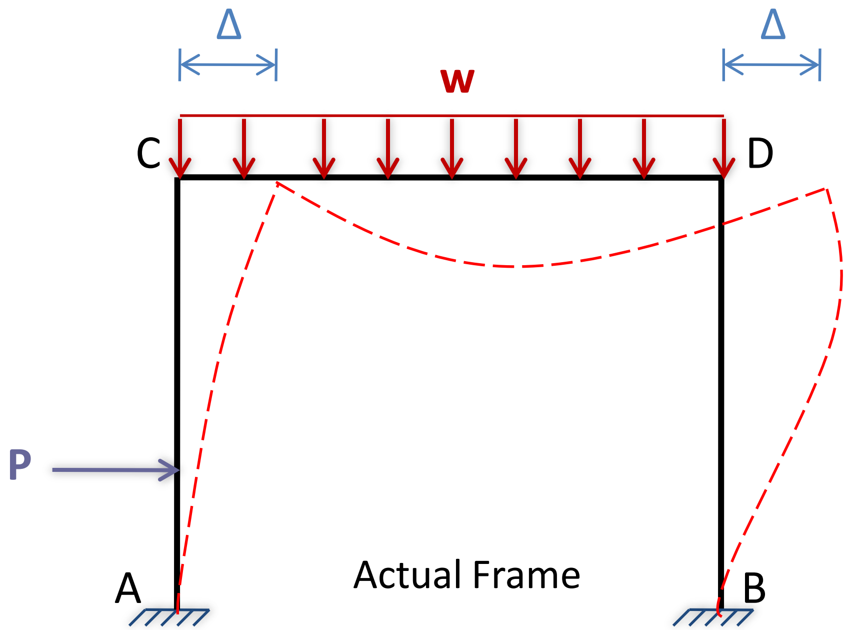
# Analysis of Frames With Sidesway

Consider the rectangular frame shown in Figure.



A qualitative deflected shape of the frame for an arbitrary loading is shown in the figure using an exaggerated scale.

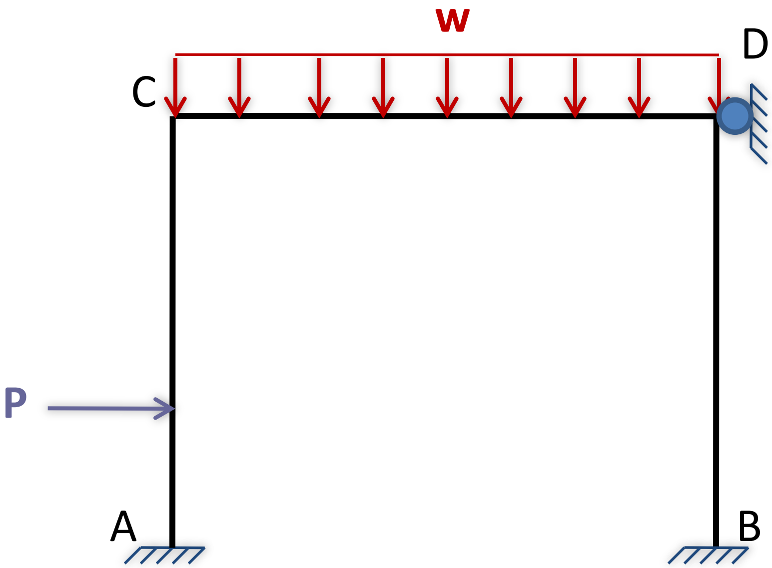
While the fixed joints A and B of the frame are completely restrained against rotation as well as translation, the joints C and D are free to rotate and translate.



Since the members of the frame are assumed to be inextensible and the deformations are assumed to be small, the joints C and D displace by the same amount Δ, in the horizontal direction only.

The MD analysis of such a frame, with sidesway, is carried out in two parts.

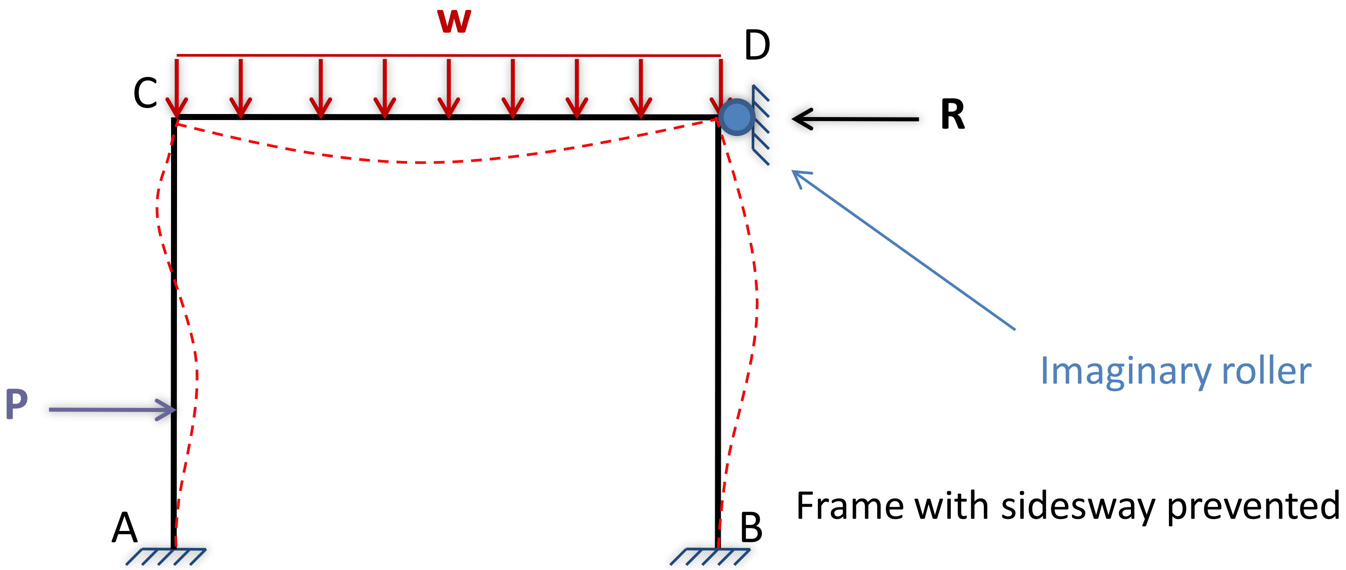
In the first part, the sidesway of the frame is prevented by adding an imaginary roller to the structure.



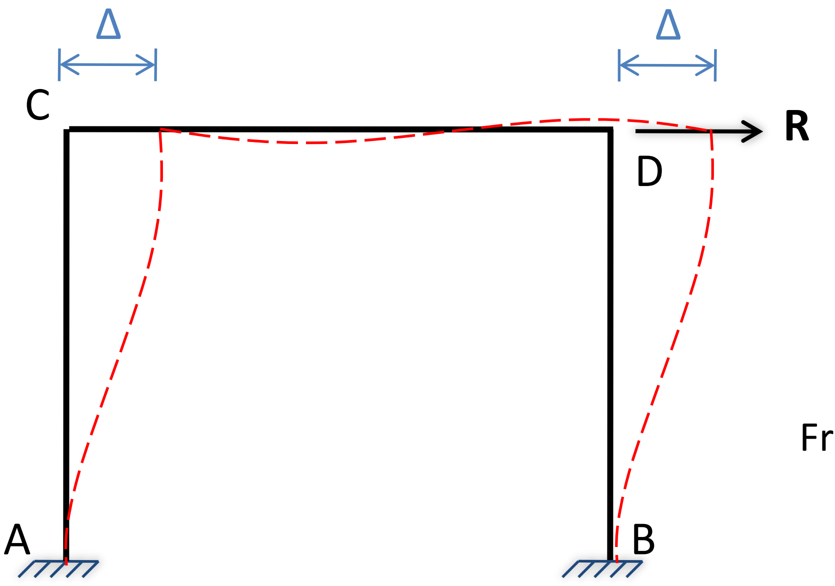
Frame with sidesway prevented

External loads are then applied to this frame, and MEM are computed by applying the MD process in the usual manner.

With the MEM known, the restraining force (reaction) R that develops at the imaginary support is evaluated by applying the equations of equilibrium.



In the second part of the analysis, the frame is subjected to the force R, which is applied in the opposite direction, as shown in the next slide.



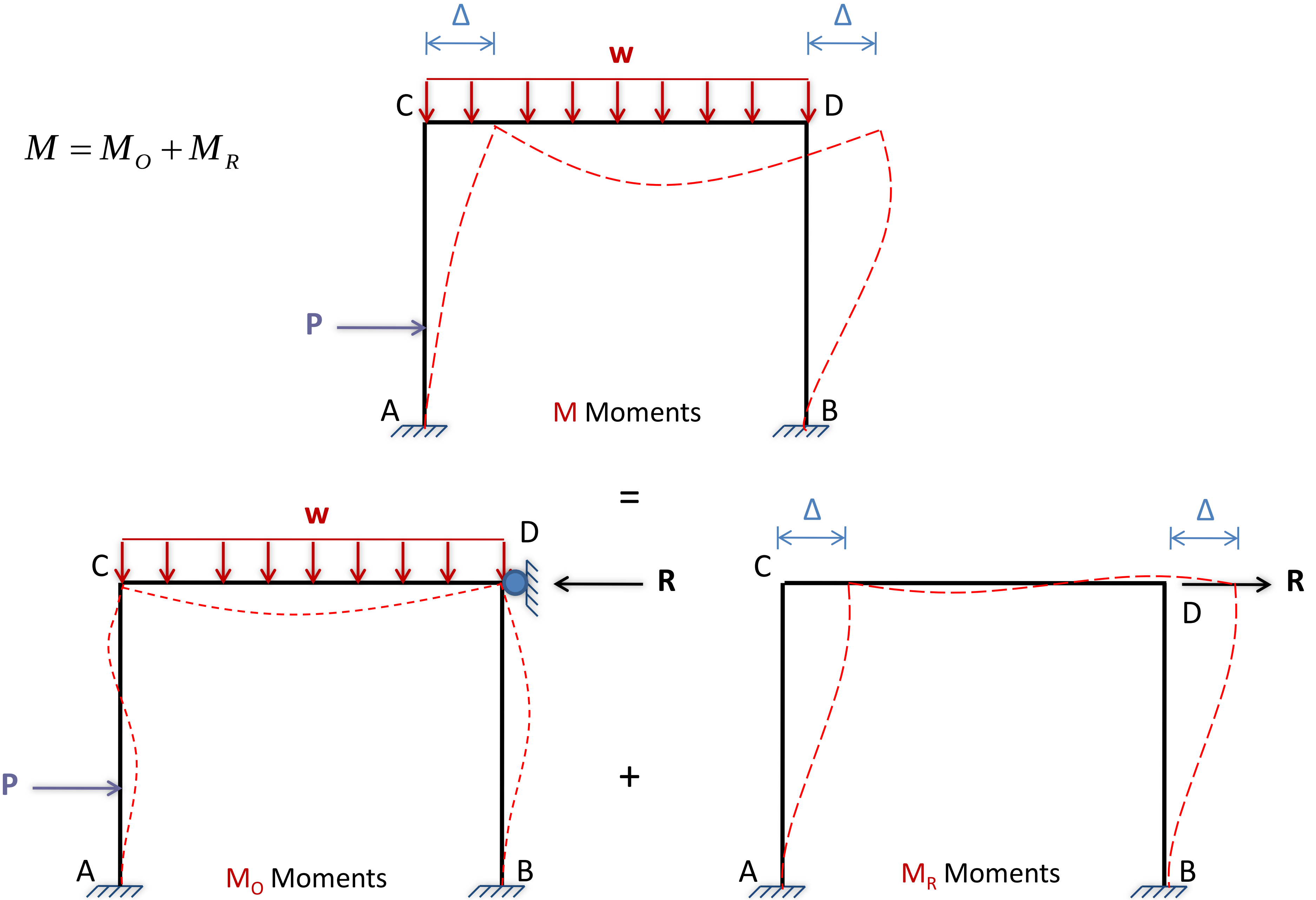
Frame subjected to R

The moments that develop at the member ends are determined and superimposed on the moments computed in the first part to obtain the member end moments in the actual frame.

If M, $M_O$, and $M_R$ denote, respectively, the MEM in the actual frame, the frame with sidesway prevented, and the frame subjected to R, then we can write

$$M = M_O + M_R$$

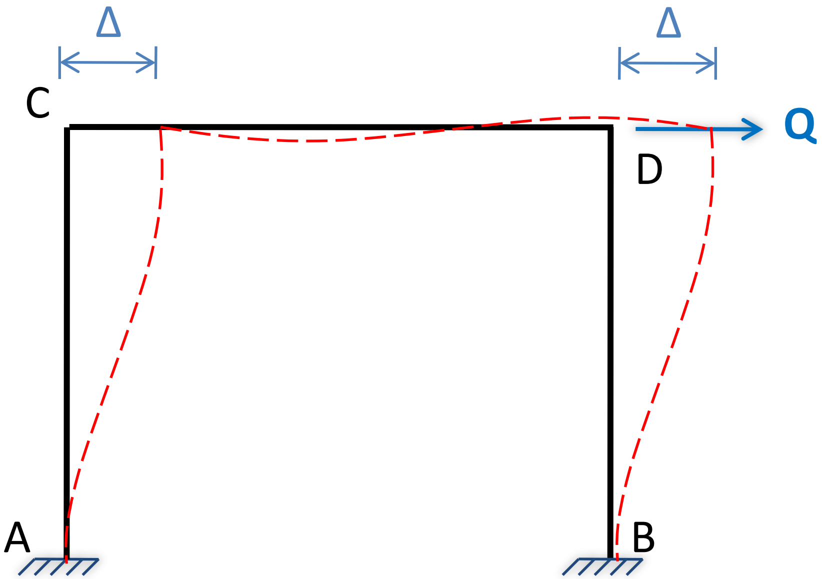
$$M = M_O + M_R$$



An important question that arises in the second part is “how to determine the member end moments $M_R$ that develop when the frame undergoes sidesway under the action of R”.

The MDM cannot be used directly to compute the moments due to the known lateral load R, we employ an indirect approach in which the frame is subjected to an arbitrary known joint translation Δ’ caused by an unload load Q acting at the location and in the direction of R.

From the known joint translation, Δ’, we determine the relative translation between the ends of each member, and we calculate the member FEMs in the same manner as done previously in the case of support settlements.



Frame subjected to an arbitrary
Translation Δ’
$M_Q$ Moments

The FEMs thus obtained are distributed by the MD process to determine the MEMs $M_Q$ caused by the yet-unknown load Q.

Once the MEMs $M_Q$ have been determined, the magnitude of Q can be evaluated by the application of equilibrium equations.

With the load Q and the corresponding moments $M_Q$ known, the desired moments $M_R$ due to the lateral load R can now be determined easily by multiplying $M_Q$ by the ratio R/Q; that is

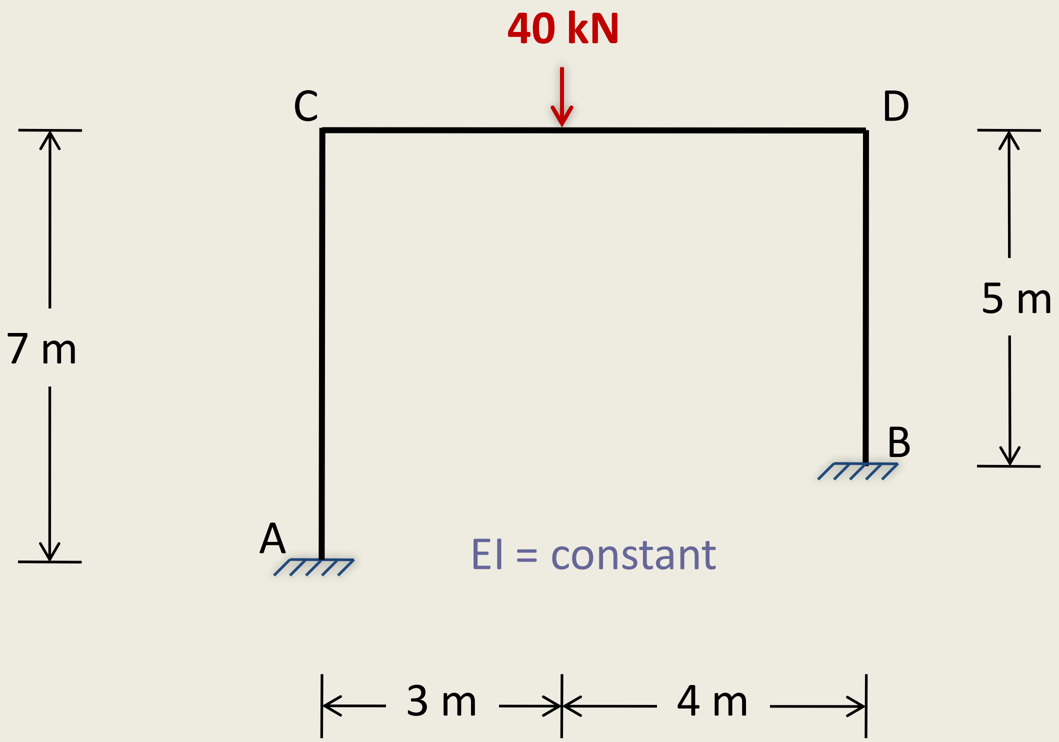
$$M_R = \left(\frac{R}{Q}\right) M_Q$$

By substituting this Equation into the last Equation, w can express the MEMs in the actual frame as

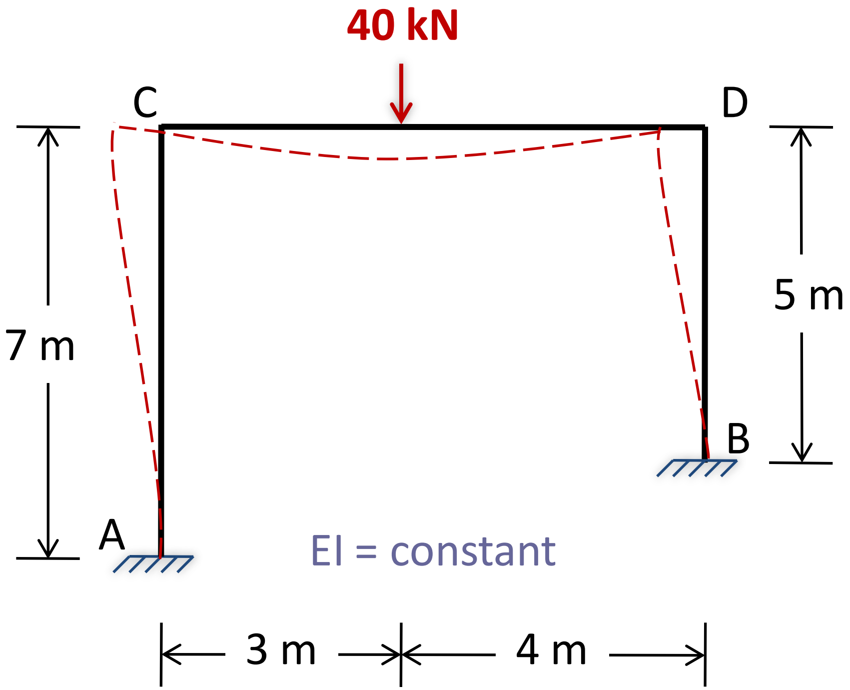
$$M = M_O + \left(\frac{R}{Q}\right) M_Q$$

# Example 2

- Determine the member end moments and reactions for the frame shown by the moment-distribution method.



# Solution



- Distribution Factors

At joint C

$$DF_{CA} = DF_{CD} = \frac{I/7}{2(I/7)} = 0.5$$

At joint D

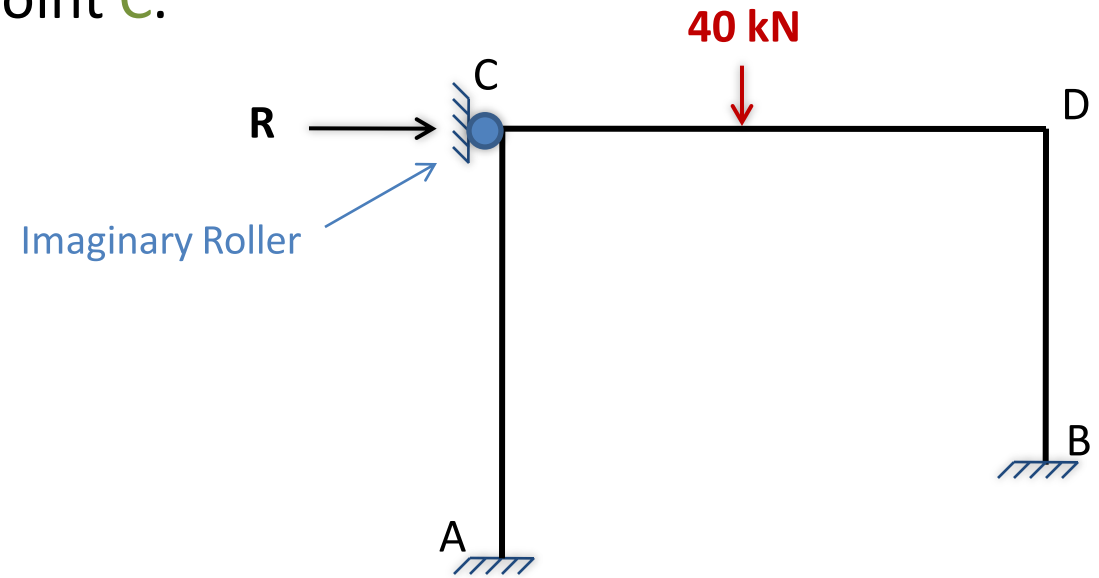
$$DF_{DC} = \frac{I/7}{(I/7)+(I/5)} = 0.417$$

$$DF_{DB} = \frac{I/5}{(I/7)+(I/5)} = 0.583$$

$$DF_{DC} + DF_{DB} = 0.417 + 0.583 = 1$$ Checks

- Part 1: Sidesway Prevented

The sidesway of frame is prevented by adding an imaginary roller at joint C.



Frame with sidesway prevented

Assuming that joint C and D of this frame are clamped against rotation, we calculate the FEMs due to the external loads to be

$$FEM_{CD} = +39.2\ \mathrm{kN.m}$$

$$FEM_{DC} = -29.4\ \mathrm{kN.m}$$

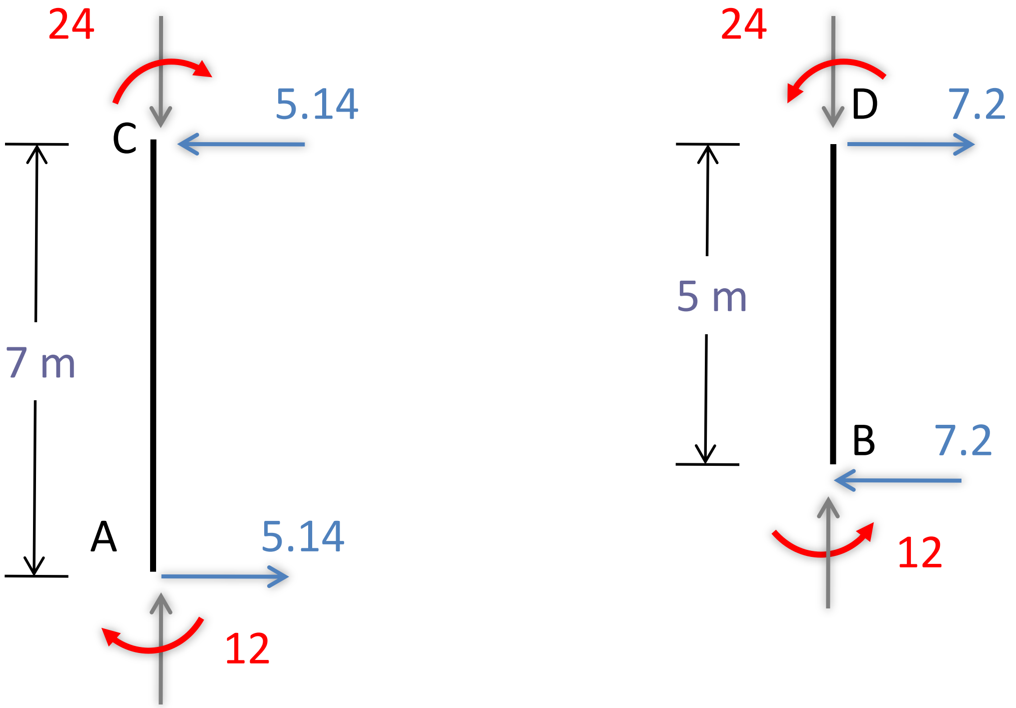
$$FEM_{AC} = FEM_{CA} = FEM_{BD} = FEM_{DB} = 0$$

The MD of these FEMs is then performed, as shown on the MD Table to determine the MEMs “$M_O$” in the frame with sidesway prevented.

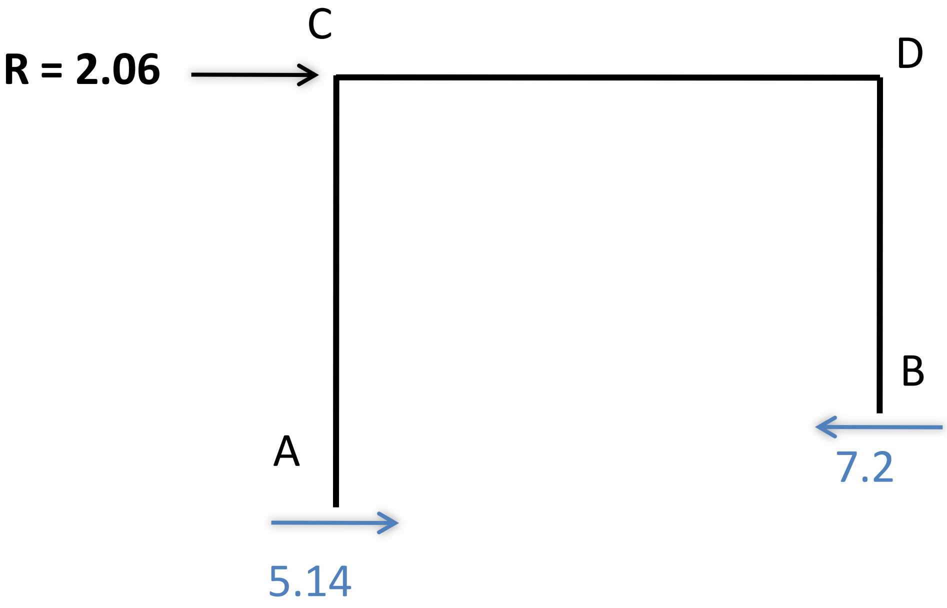
| **AC** | **CA** | **CD** | **DC** | **DB** | **BD** |
|---|---|---|---|---|---|
| | 0.5 | 0.5 | 0.417 | 0.583 | |
| | | +39.2 | -29.4 | | |
| | -19.6 | -19.6 | +12.3 | +17.1 | |
| -9.8 | | +6.2 | -9.8 | | +8.6 |
| | -3.1 | -3.1 | +4.1 | +5.7 | |
| -1.6 | | +2.1 | -1.6 | | +2.9 |
| | -1.1 | -1.1 | +0.7 | +0.9 | |
| -0.6 | | +0.4 | -0.6 | | +0.5 |
| | -0.2 | -0.2 | +0.3 | +0.3 | |
| **-12** | **-24** | **+23.9** | **-24** | **+24** | **+12** |

Member End Moments for Frame with Sidesway Prevented - $M_O$

To evaluate the restraining force R that develops at the imaginary roller support, we first calculate the shears at the lower ends of the columns AC and BD by considering the moment equilibrium of the free bodies of the columns.



Next, by considering the equilibrium of the horizontal forces acting on the entire frame, we determine the restraining force R to be

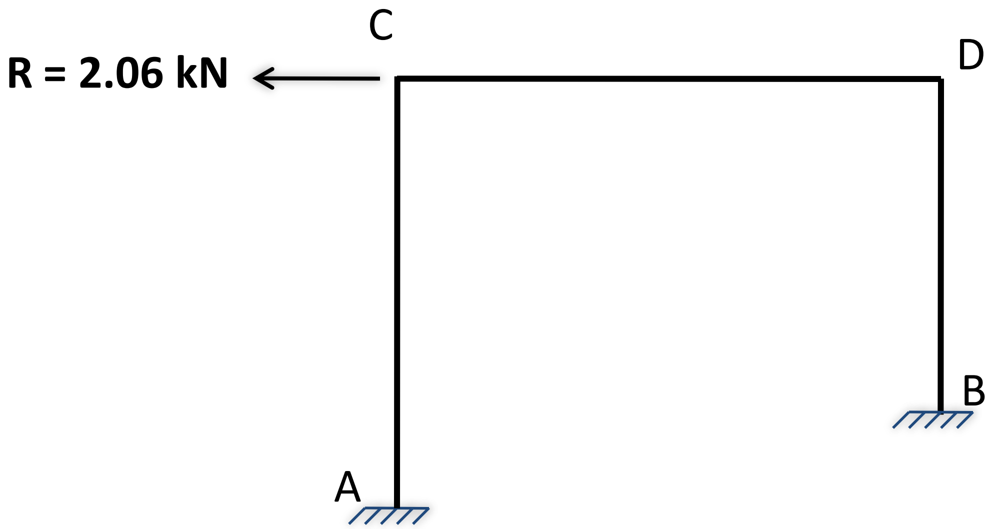


$$+\rightarrow\sum F_x = 0 \qquad R + 5.14 - 7.2 = 0 \qquad R = 2.06\,kN \rightarrow$$

Restraining force acts to the right, indicating that if the roller would not have been in place, the frame would have swayed to the left.

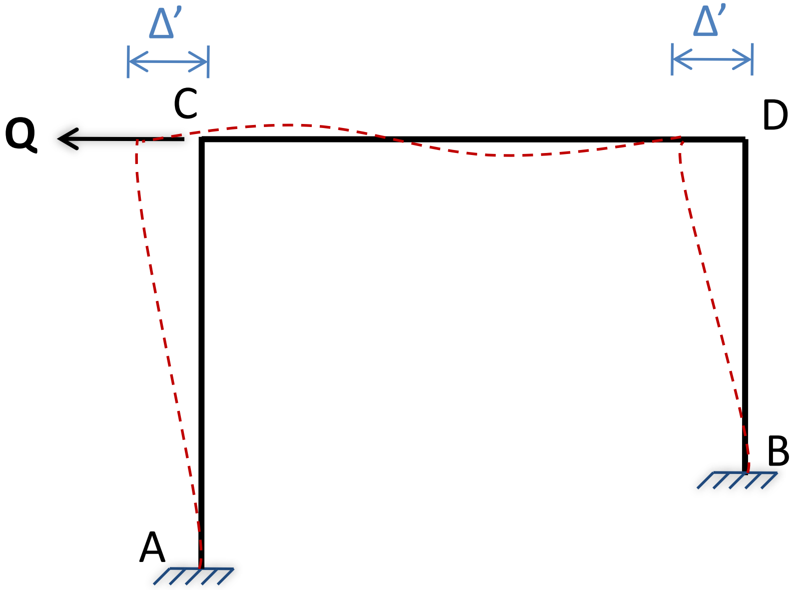
- Part 2: Sidesway Permitted

Since the actual frame is not supported by a roller at joint C, we neutralize the effect of the restraining force by applying a lateral load R = 2.06 kN in the opposite direction to the frame.



Frame subjected to R = 2.06 kN
$M_R$ Moments

MD method cannot be used directly to compute MEMs $M_R$ due to the lateral load R = 2.06 kN, we use an indirect approach in which the frame is subjected to an arbitrary known joint translation Δ’ caused by an unknown load Q acting at the location in the direction of R.



Frame subjected to an Arbitrary Translation Δ’
$M_Q$ Moments

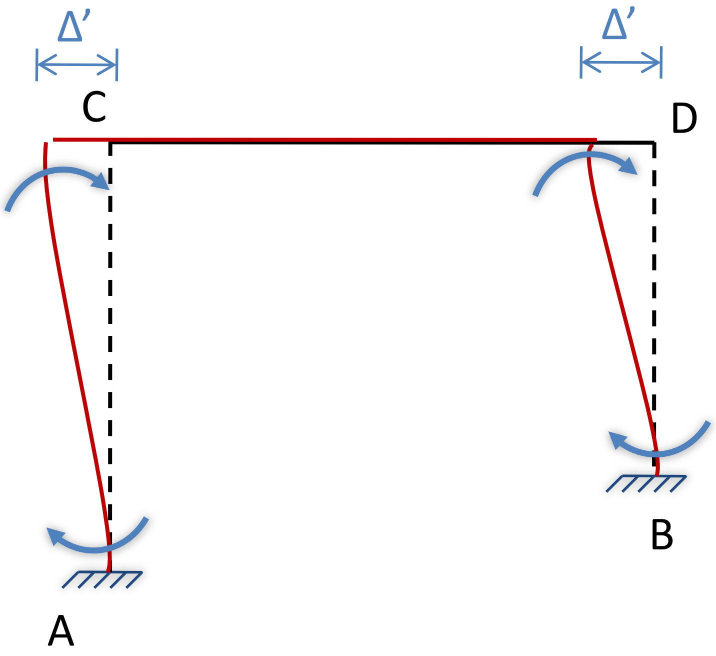
Assuming that the joints C and D of the frame are clamped against rotation as shown in figure on next slide, FEMs due to the translation Δ’ are given by

$$FEM_{AC} = FEM_{CA} = \frac{-6EI\Delta'}{(7)^2} = \frac{-6EI\Delta'}{49}$$

$$FEM_{BD} = FEM_{DB} = \frac{-6EI\Delta'}{(5)^2} = \frac{-6EI\Delta'}{25}$$

$$FEM_{CD} = FEM_{DC} = 0$$

In which negative sign have been assign to the FEMs for the columns, because these moments must act in the clockwise direction, as shown.



FEMs due to Known Translation Δ’

Instead of arbitrarily assuming a numerical value for Δ’ to compute the FEMs, it is usually more convenient to assume a numerical value for one of the FEMs, evaluate Δ’ from the expression of that FEM, and use the value of Δ’ to compute the remaining FEMs.

Thus, we arbitrarily assume the $FEM_{AC}$ to be -50 kN.m

$$FEM_{AC} = FEM_{CA} = \frac{-6EI\Delta'}{49} = -50\,\text{kN.m}$$

by solving for Δ’, we obtain

$$\Delta' = \frac{408.33}{EI}$$

by substituting this value of Δ’ into the expressions for $FEM_{BD}$ and $FEM_{DB}$, we determine the consistent values of these moments to be

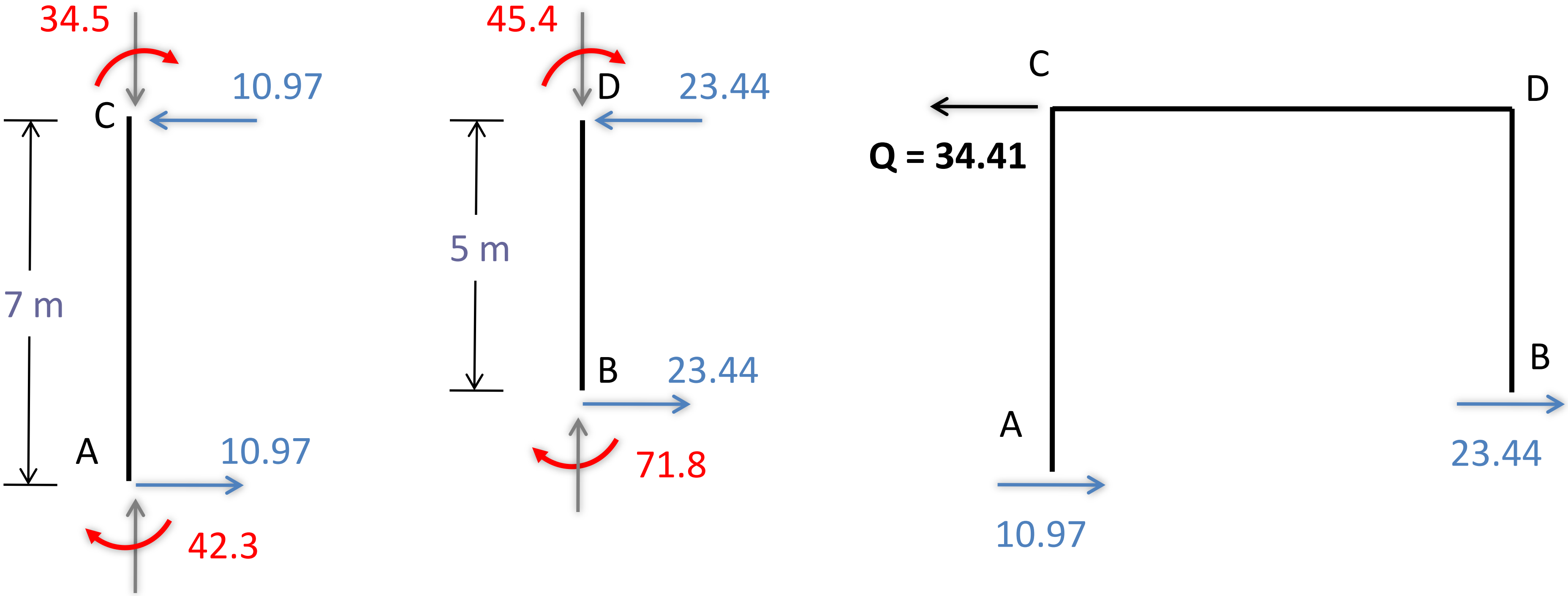
$$FEM_{BD} = FEM_{DB} = \frac{-6(408.33)}{25} = -98\text{kN.m}$$

These FEMs are then distributed by the usual MD process, to determine the MEMs $M_Q$ caused by the yet unknown load Q.

| AC | CA | CD | DC | DB | BD |
|---|---|---|---|---|---|
|  | 0.5 | 0.5 | 0.417 | 0.583 |  |
| -50 | -50 |  |  | -98 | -98 |
|  | +25 | +25 | +40.9 | +57.1 |  |
| +12.5 |  | +20.5 | +12.5 |  | +28.6 |
|  | -10.3 | -10.3 | -5.2 | -7.3 |  |
| -5.2 |  | -2.6 | -5.2 |  | -3.7 |
|  | +1.3 | +1.3 | +2.2 | +3 |  |
| +0.7 |  | +1.1 | +0.7 |  | +1.5 |
|  | -0.6 | -0.6 | -0.3 | -0.4 |  |
| -0.3 |  | -0.2 | -0.3 |  | -0.2 |
|  | +0.1 | +0.1 | +0.1 | +0.2 |  |
| **-42.3** | **-34.5** | **+34.3** | **+45.4** | **-45.4** | **-71.8** |

Member End Moments Due to Known Translation Δ’- $M_Q$

To evaluate the magnitude of Q that corresponds to these MEMs, we first calculate shears at the lower ends of the columns by considering their moment equilibrium and then apply the equation of equilibrium in the horizontal direction to the entire frame



$$+\rightarrow\sum F_x = 0 \qquad -Q + 10.97 + 23.44 = 0 \qquad Q = 34.41\,kN \leftarrow$$

which indicates that the moments $M_Q$ computed are caused by a lateral load $Q = 34.41$ kN.

Since the moments are linearly proportional to the magnitude of the load, the desired moment $M_R$ due to the lateral load $R = 2.06$ kN must be equal to the moment $M_Q$ multiplied by the ratio $R/Q = 2.06/34.41$.

- Actual Member End Moments

The actual MEMs , M, can now be determined by algebraically summing the MEMs $M_O$ and 2.06/34.41 times the MEMs $M_Q$.

$$M_{AC} = -12 + \left(\frac{2.06}{34.41}\right)(-42.3) = -14.5\,\text{kN}\,.\,\text{m} \qquad ANS$$

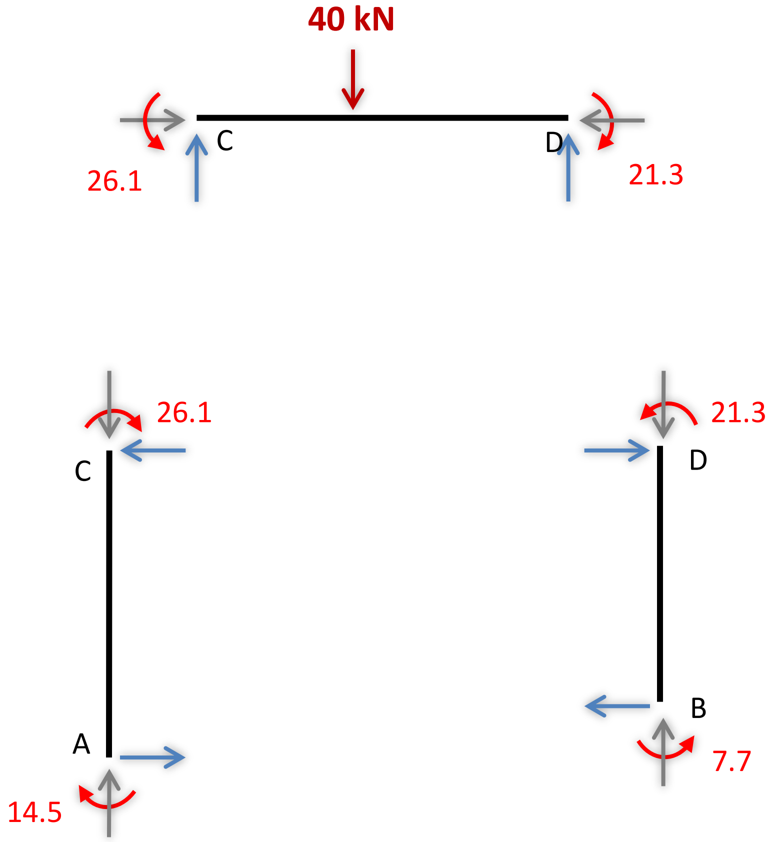
$$M_{CA} = -24 + \left(\frac{2.06}{34.41}\right)(-34.5) = -26.1\,\text{kN}\,.\,\text{m} \qquad ANS$$

$$M_{CD} = 23.9 + \left(\frac{2.06}{34.41}\right)(34.3) = 26\,\text{kN}\,.\,\text{m} \qquad ANS$$

$$M_{DC} = -24 + \left(\frac{2.06}{34.41}\right)(45.4) = -21.3\,\text{kN}\,.\,\text{m} \qquad ANS$$

$$M_{DB} = 24 + \left(\frac{2.06}{34.41}\right)(-45.4) = 21.3\,\text{kN}\,.\,\text{m} \qquad ANS$$

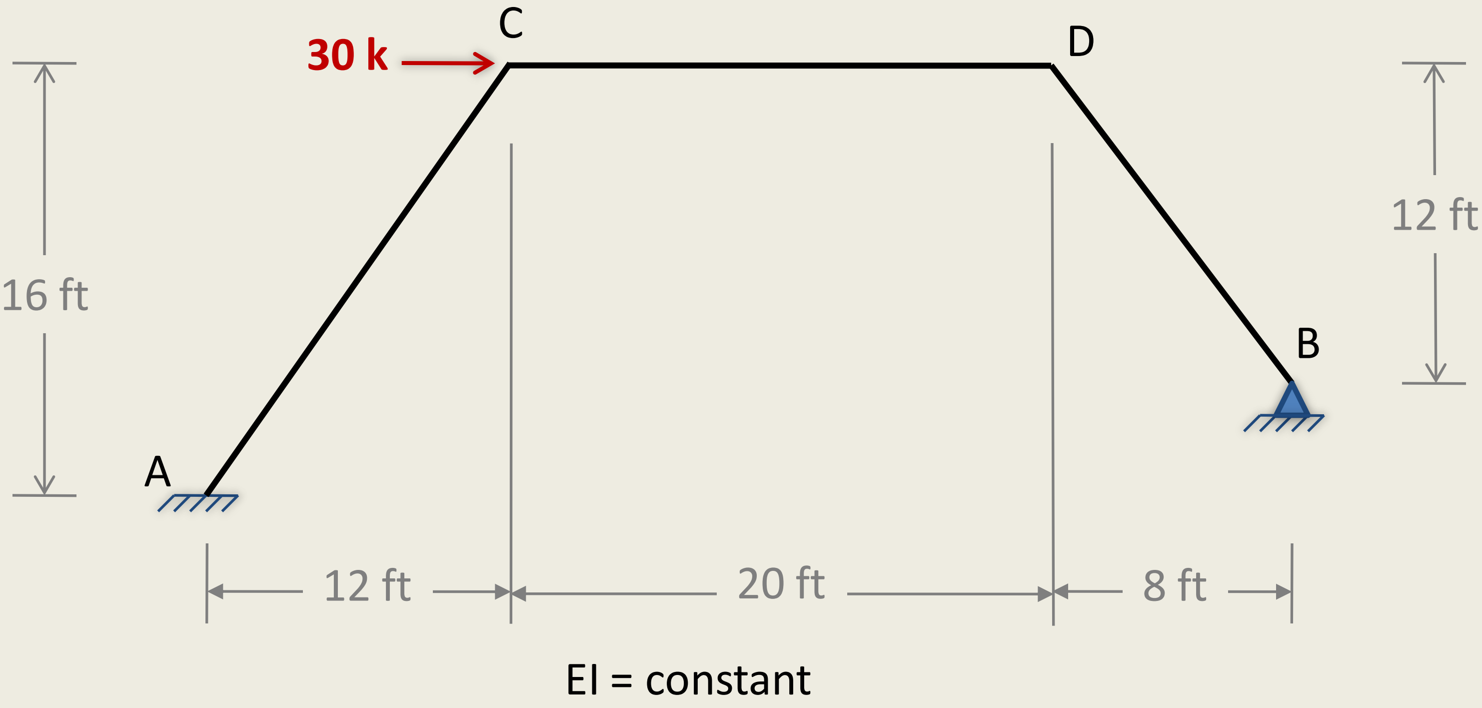
$$M_{BD} = 12 + \left(\frac{2.06}{34.41}\right)(-71.8) = 7.7\,\text{kN}\,.\,\text{m} \qquad ANS$$



Actual Member End Moments (kN . m)

# Example 3

- Determine the member end moments and reactions for the frame shown by the moment-distribution method.

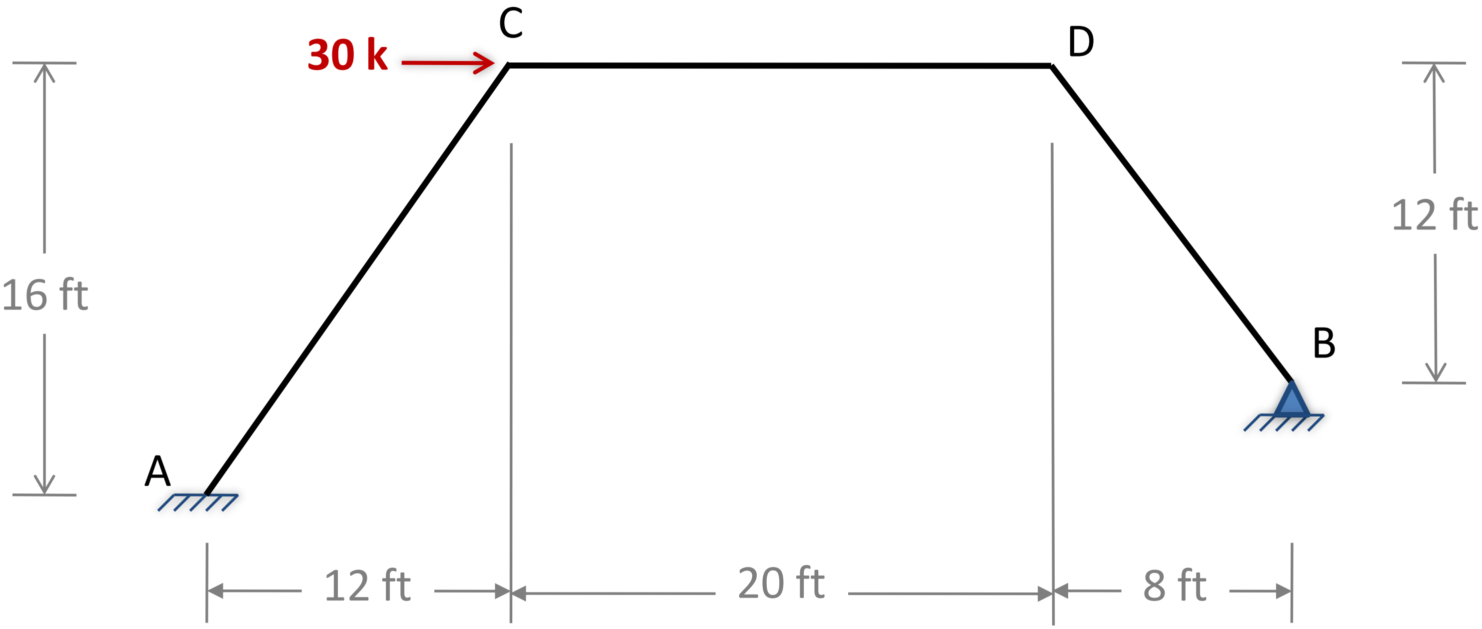


# Solution

- Distribution Factors

At joint C

$$DF_{CA} = DF_{CD} = \frac{\frac{I}{20}}{2\left(\frac{I}{20}\right)} = 0.5$$



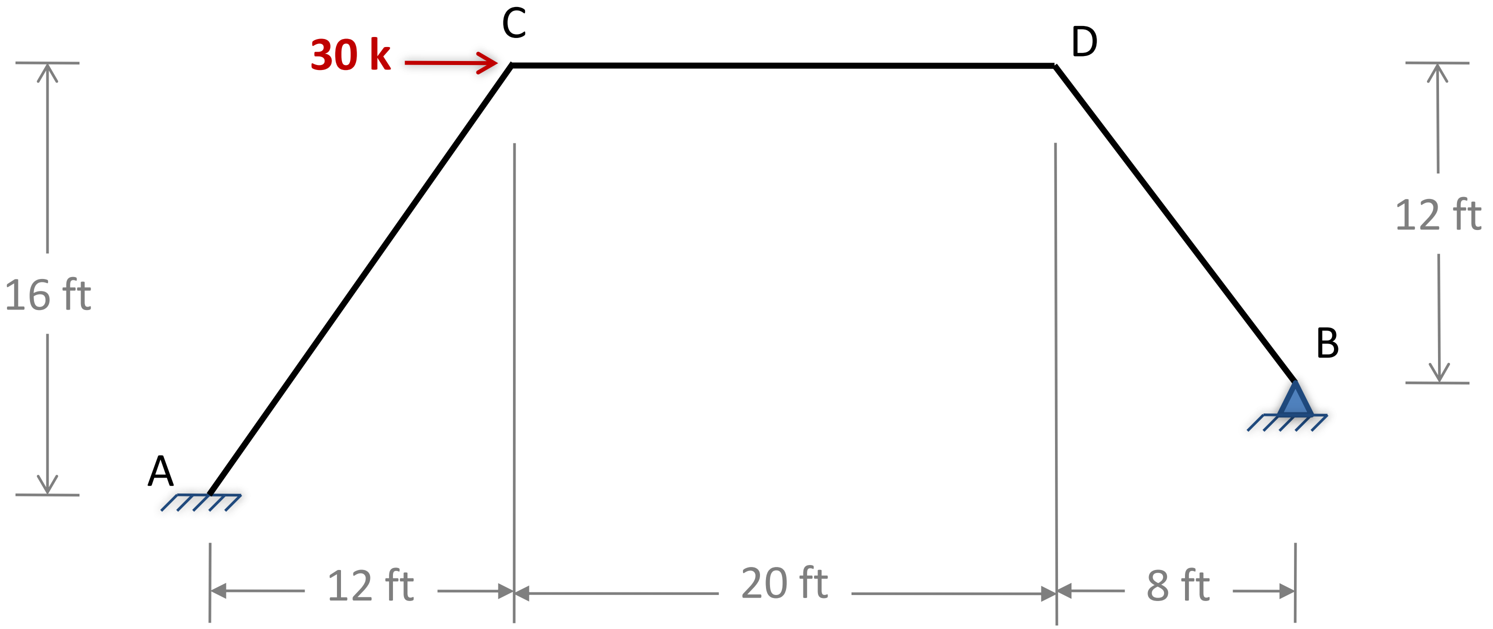
EI = constant

# Solution

- Distribution Factors

At joint D

$$DF_{DC} = \frac{\frac{I}{20}}{\left(\frac{I}{20}\right)+\left(\frac{3}{4}\right)\left(\frac{I}{14.42}\right)} = 0.49 \qquad DF_{DB} = \frac{\left(\frac{3}{4}\right)\left(\frac{I}{14.42}\right)}{\left(\frac{I}{20}\right)+\left(\frac{3}{4}\right)\left(\frac{I}{14.42}\right)} = 0.51$$



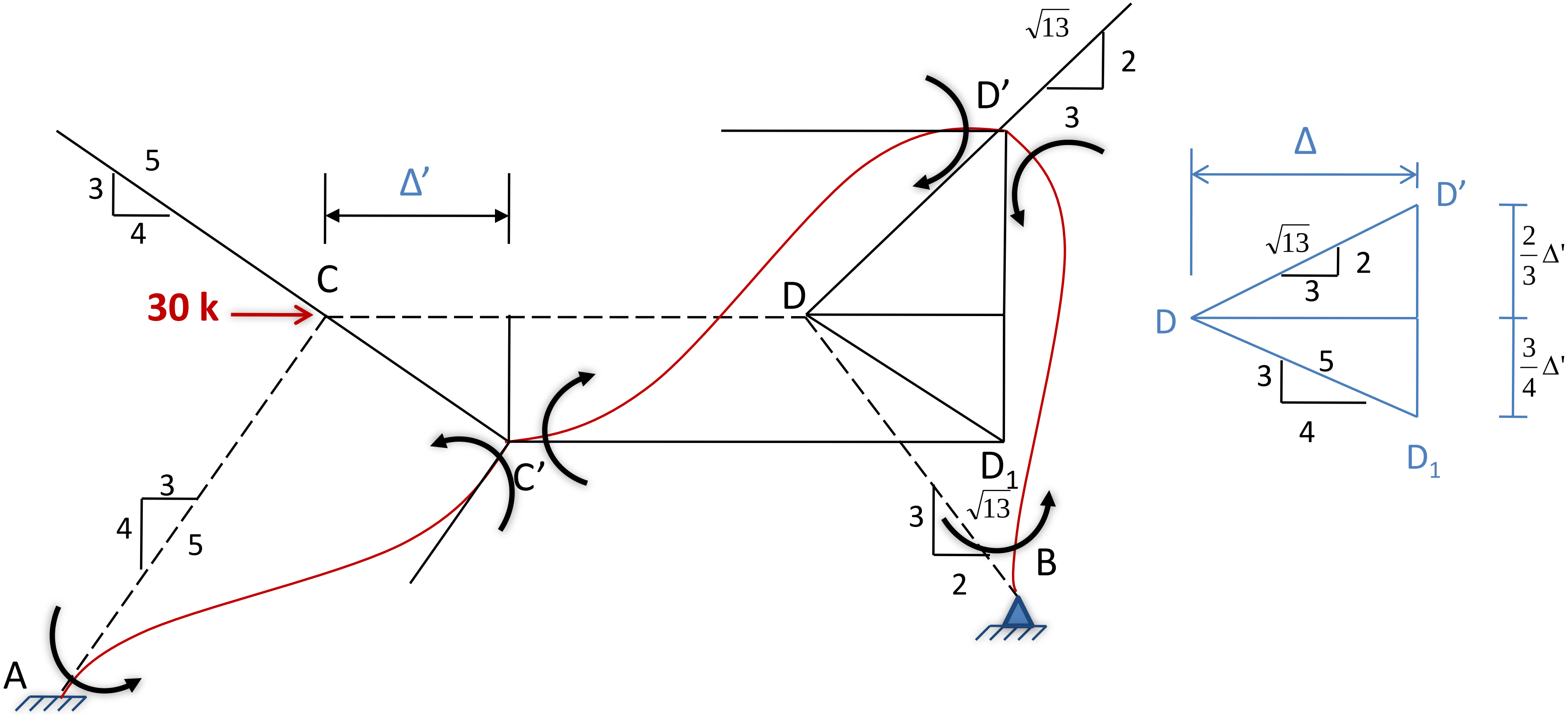
EI = constant

## MEMs due to an Arbitrary Sidesway Δ’

Since no external loads are applied to the members of the frame, the MEMs $M_O$ in the frame restrained against sidesway will be zero.

To determine the MEMs M due to the 30-k lateral load, we subject the frame to an arbitrary known horizontal translation Δ’ at joint C.

Figure on the next slide shows a qualitative deflected shape of the frame with all joints clamped against rotation and subjected to the horizontal displacement Δ’ at joint C.



FEMs due to an Arbitrary Translation Δ’

# MEMs due to an Arbitrary Sidesway Δ’

Note that, since the frame members are assumed to be inextensible and deformations are assumed to be small, an end of a member can translate only in a direction perpendicular to the member.

From this figure, we can see that the relative translation $\Delta_{AC}$ between the ends of members AC in the direction perpendicular to the member can be expressed in terms of the joint translation Δ’ as

$$\Delta_{AC} = CC' = \frac{5}{4}\Delta' = 1.25\Delta'$$

# MEMs due to an Arbitrary Sidesway Δ’

Similarly, the relative translation for members CD and BD are given by

$$\Delta_{CD} = D_1D' = \frac{2}{3}\Delta' + \frac{3}{4}\Delta' = 1.417\Delta'$$

$$\Delta_{BD} = DD' = \frac{\sqrt{13}}{3}\Delta' = 1.202\Delta'$$

The FEMs due to the relative translation are

$$FEM_{AC} = FEM_{CA} = \frac{6EI(1.25\Delta')}{(20)^2}$$

$$FEM_{CD} = FEM_{DC} = -\frac{6EI(1.417\Delta')}{(20)^2}$$

$$FEM_{BD} = FEM_{DB} = \frac{6EI(1.202\Delta')}{(14.42)^2}$$

# MEMs due to an Arbitrary Sidesway Δ’

in which the FEMs for members AC and BD are CCW (positive), whereas those for member CD are CW (negative).

If we arbitrarily assume that

$$FEM_{BD} = FEM_{DB} = \frac{6EI\left(1.202\Delta'\right)}{\left(14.42\right)^2} = 100\,\mathrm{k\text{-}ft}$$

then

$$EI\Delta' = 2883.2$$

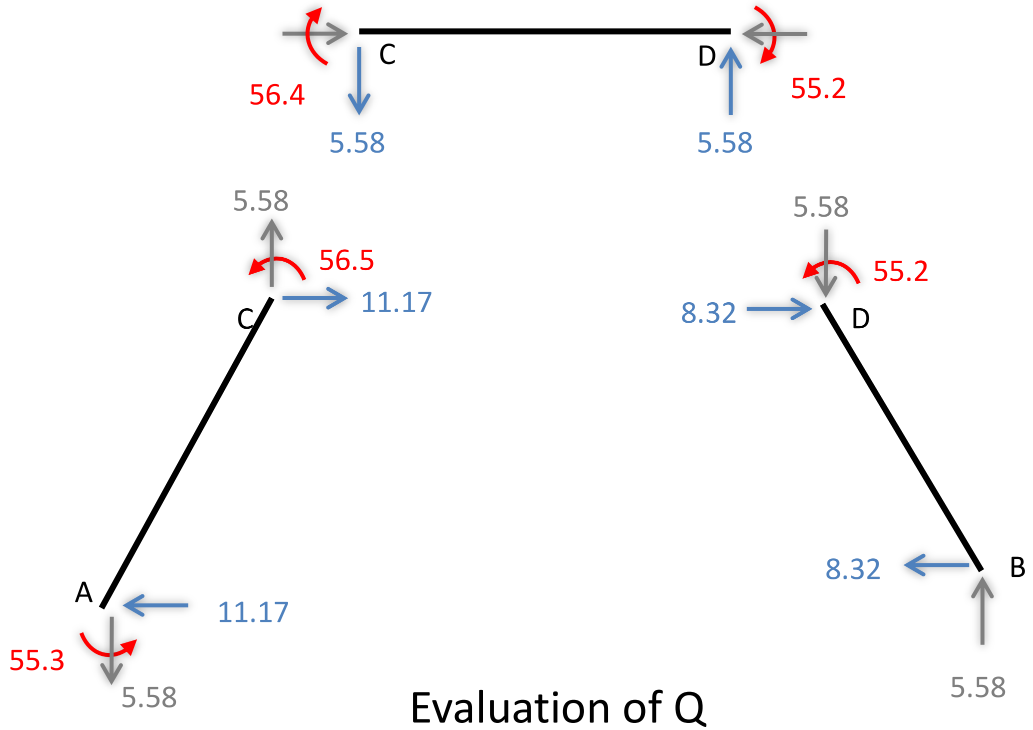
and therefore

$$FEM_{AC} = FEM_{CA} = 54.1\,\mathrm{k\text{-}ft}$$

$$FEM_{CD} = FEM_{DC} = -61.3\,\mathrm{k\text{-}ft}$$

The FEMs are distributed by the MD process to determine the MEMs $M_Q$.

To determine the magnitude of the load Q that corresponds to the MEMs $M_Q$ we first calculate the shears at the ends of the girder CD by considering the moment equilibrium of the free body of the girder as shown.



Evaluation of Q

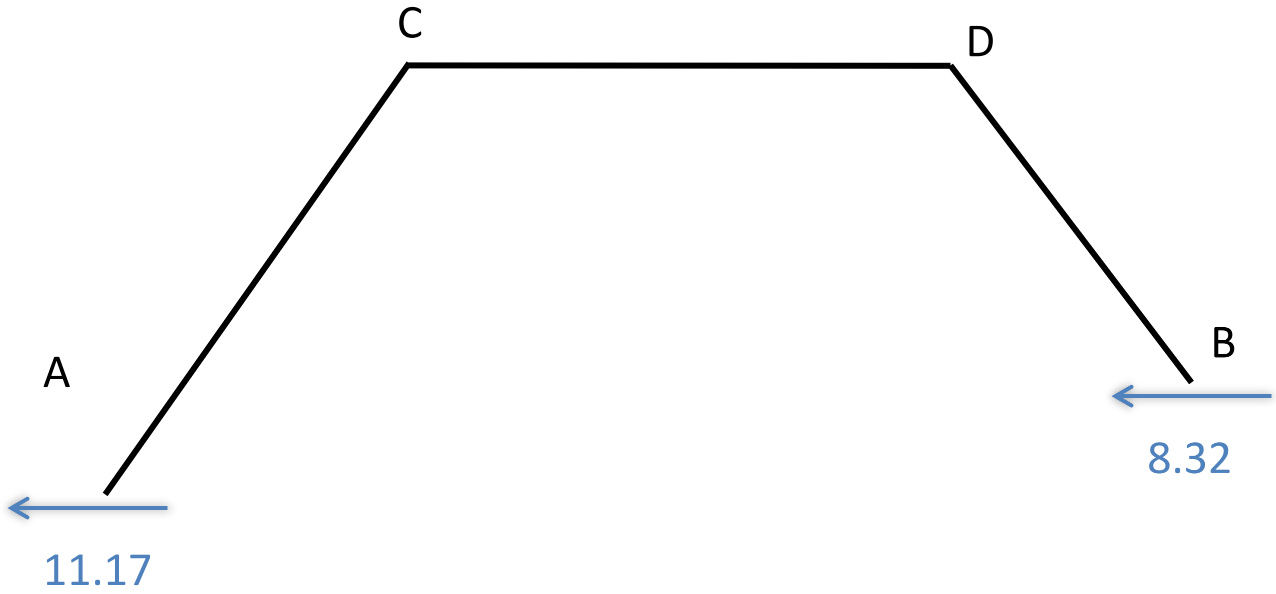
## MD TABLE

| AC | CA | CD | DC | DB | BD |
|---|---|---|---|---|---|
| | 0.5 | 0.5 | 0.49 | 0.51 | 1 |
| +54.1 | +54.1 | -61.3 | -61.3 | +100 | +100 |
| | +3.6 | +3.6 | -19 | -19.7 | -100 |
| +1.8 | | -9.5 | +1.8 | -50 | |
| | +4.8 | +4.8 | +23.6 | +24.6 | |
| +2.4 | | +11.8 | +2.4 | | |
| | -5.9 | -5.9 | -1.2 | -1.2 | |
| -3 | | -0.6 | -3 | | |
| | +0.3 | +0.3 | +1.5 | +1.5 | |
| +0.2 | | +0.8 | +0.2 | | |
| | -0.4 | -0.4 | -0.1 | -0.1 | |
| -0.2 | | | -0.2 | | |
| | | | +0.1 | +0.1 | |
| **+55.3** | **+56.5** | **-56.4** | **-55.2** | **+55.2** | **0** |

Member End Moments Due to Known Translation Δ’- $M_Q$

The girder shears (5.58 k) thus obtained are then applied to the free bodies of the inclined members AC and BD.

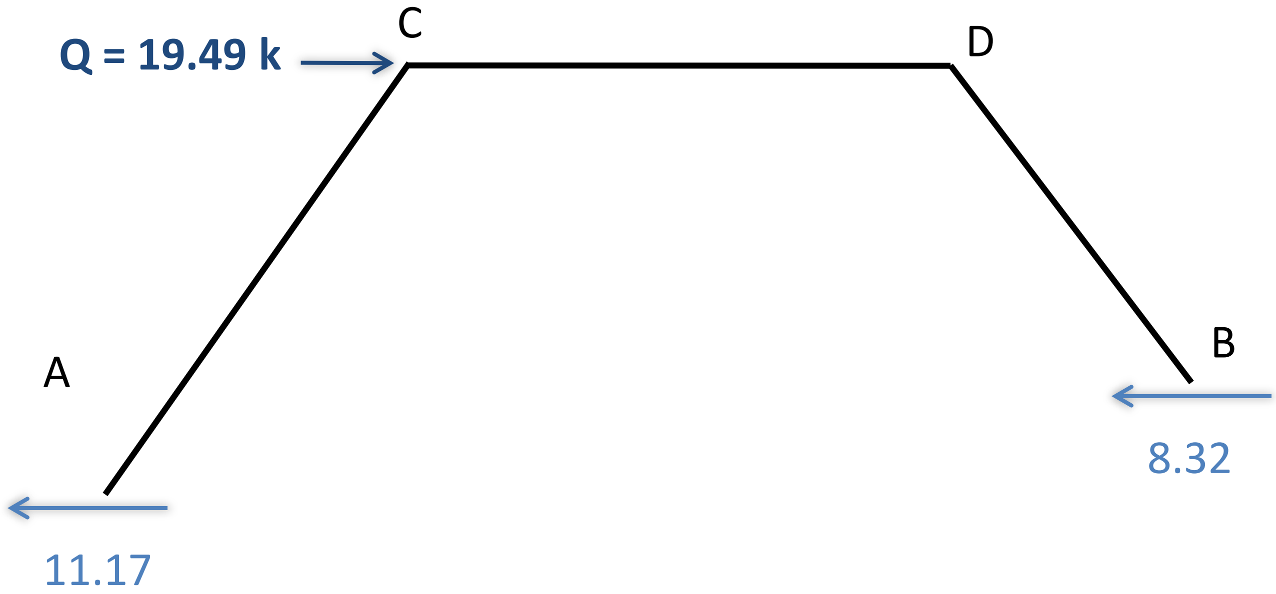
Next, we apply the equations of moment equilibrium to members AC and BD to calculate the horizontal forces at the lower ends of these members.



Evaluation of Q

The magnitude of Q can now be determined by considering the equilibrium of horizontal forces acting on the entire frame as shown below

$$+\rightarrow \sum F_x = 0 \qquad Q - 11.17 - 8.32 = 0 \qquad Q = 19.49\,\mathrm{k} \leftarrow$$



Evaluation of Q

## Actual MEMs

The actual MEMs, M, due to the 30-k lateral load can now be evaluated by multiplying the moments $M_Q$ computed in Table by the ratio 30/Q=30/19.49:

$$M_{AC} = \frac{30}{19.49}(55.3) = 85.1\,\text{k-ft} \qquad \text{ANS}$$

$$M_{CA} = \frac{30}{19.49}(56.5) = 87\,\text{k-ft} \qquad \text{ANS}$$

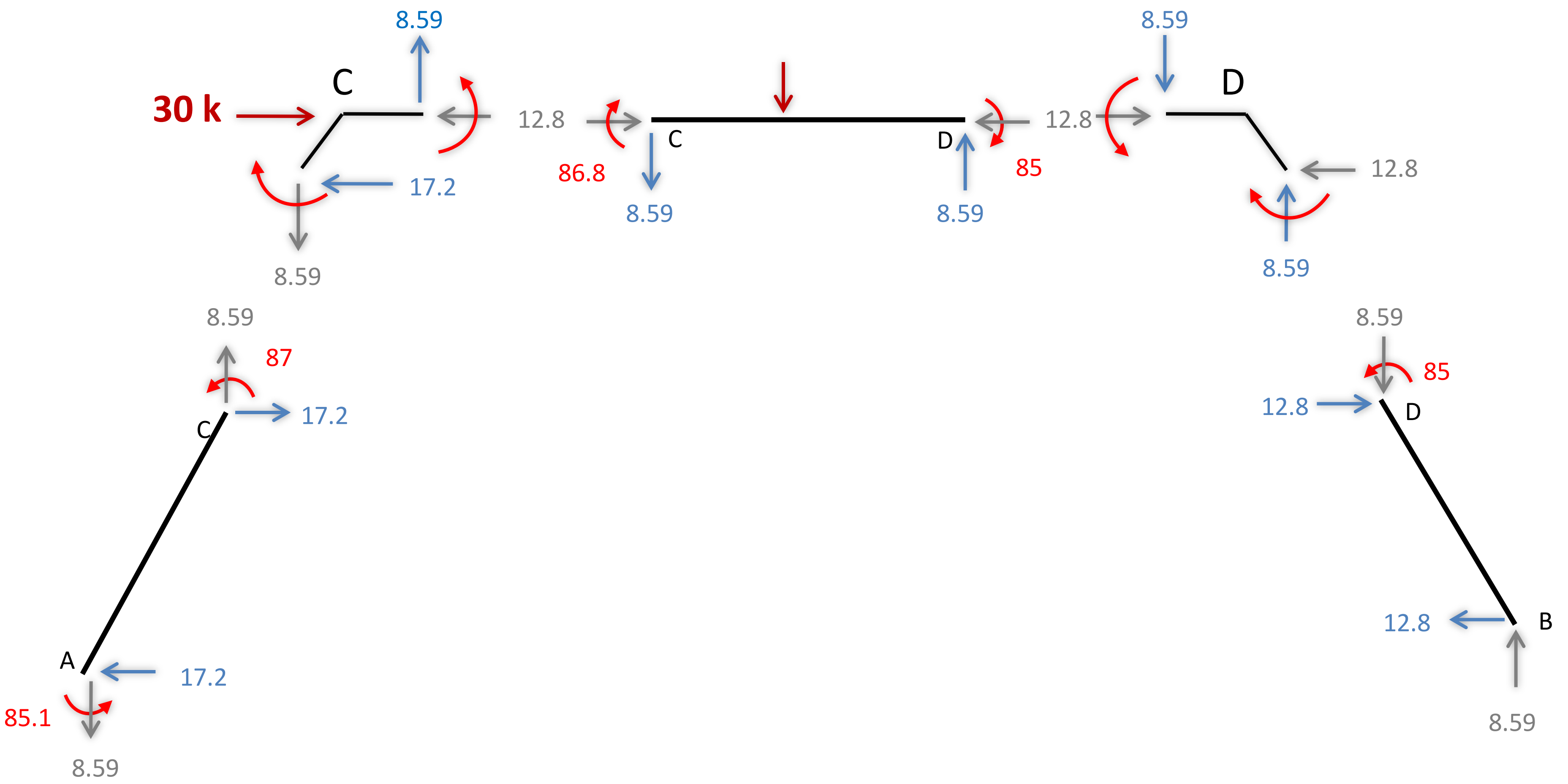
$$M_{CD} = \frac{30}{19.49}(-56.4) = -86.8\,\text{k-ft} \qquad \text{ANS}$$

$$M_{DC} = \frac{30}{19.49}(-55.2) = -85\,\text{k-ft} \qquad \text{ANS}$$

$$M_{DB} = \frac{30}{19.49}(55.2) = 85\,\text{k-ft} \qquad \text{ANS}$$

$$M_{BD} = 0 \qquad \text{ANS}$$

# Member End Forces



Evaluation of Q

# Support Reactions