Method of Least Work

Theory of Structures-II
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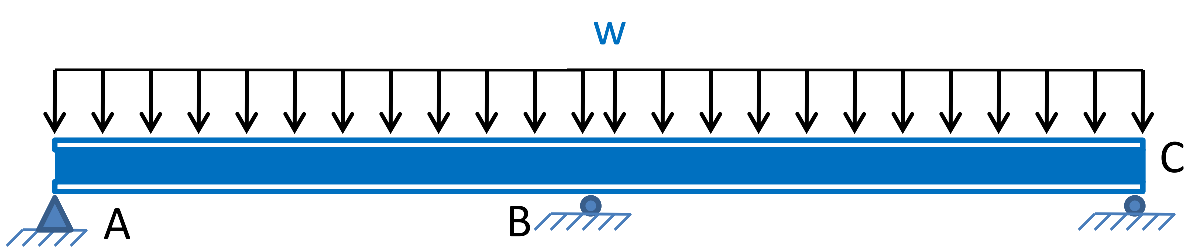
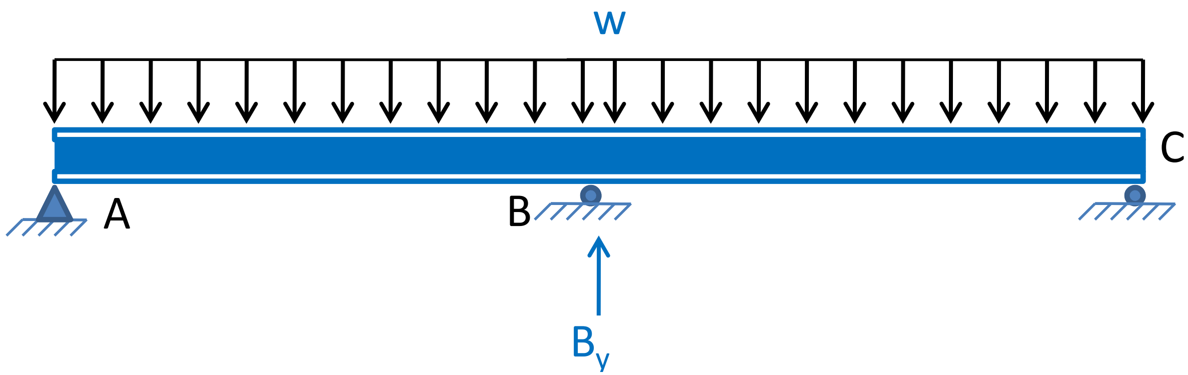
Method of Least Work / Castigliano's Second Theorem

- Force Method
- Compatibility equations are established by using the Castigliano's second theorem, instead of by deflection superposition as in method of consistent deformations.
- Let us consider a statically indeterminate beam with unyielding supports subjected to an external loading w.



Method of Least Work / Castigliano's Second Theorem



- Suppose that we select the vertical reaction B_y at the interior support B to be the redundant.
- By treating the redundant as an unknown load applied to the beam along with the prescribed loading w, an expression for the strain energy can be written in terms of known load w and the unknown redundant B_y as

$$U = f\left(w, B_y\right)$$

Method of Least Work / Castigliano's Second Theorem

- Above equation indicates symbolically that the strain energy for the beam is expressed as a function of the known external load w and the unknown redundant B_y.

Castigliano's second theorem

"The partial derivative of the strain energy with respect to a force equals the deflection of the point of the application of the force along its line of action".

Method of Least Work / Castigliano's Second Theorem

- Since the deflection at the point of application of the redundant B_y is zero, by applying the Castigliano's second theorem, we can write

$$\frac{\partial U}{\partial B_y} = 0$$

- It should be realize that this equation represents the compatibility equation in the direction of redundant B_y, and it can be solved for the redundant.
- This equation states that the first partial derivative of the strain energy with respect to the redundant must be equal to zero.

Method of Least Work / Castigliano's Second Theorem

- This implies that for the value of the redundant that satisfies the equations of equilibrium and compatibility, the strain energy of the structure is a minimum or maximum.

- Since for a linearly elastic, there is no maximum value of strain energy, because it can be increased indefinitely by increasing the value of the redundant, we conclude that for the true value of the redundant the strain energy must be a minimum.

Method of Least Work / Castigliano's Second Theorem

- This conclusion is known as Principle of Least Work.

  "*The magnitudes of the redundants of a statically indeterminate structure must be such that the strain energy stored in the structure is a minimum (i.e., the internal work done is the least).*"

- If a structure is indeterminate to the nth degree, the n redundants are selected, and the strain energy for the structure is expressed in terms of the known external loading and the n unknown redundants as

Method of Least Work / Castigliano's Second Theorem

- If a structure is indeterminate to the nth degree, the n redundants are selected, and the strain energy for the structure is expressed in terms of the known external loading and the n unknown redundants as

$$U = f\left(w, R_1, R_2, R_3, \ldots, R_n\right)$$

in which w represents all the known loads and R_1, R_2,…, R_n denote the n redundants.

Method of Least Work / Castigliano's Second Theorem

- Next, the principle of least work is applied separately for each redundant by partially differentiating the strain energy expressions with respect to each of the redundants and by setting each partial derivative equal to zero; that is,

$$\frac{\partial U}{\partial R_1} = 0, \frac{\partial U}{\partial R_2} = 0, \cdots, \frac{\partial U}{\partial R_n} = 0$$

which represents a system of n simultaneous equations in terms of n redundants and can be solved for the redundants.

Method of Least Work / Castigliano's Second Theorem

- The strain energy of a beam subjected only to bending can be expressed as

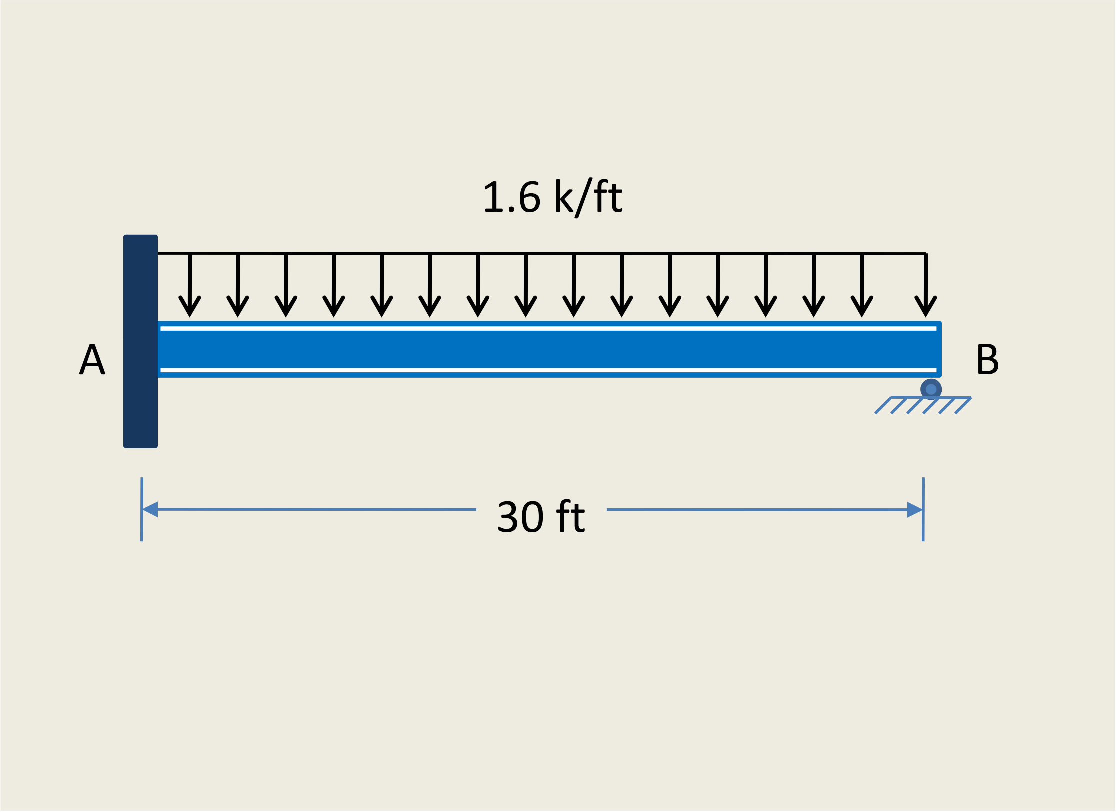
$$U = \int_0^L \frac{M^2}{2EI} dx \qquad (1)$$

- According to the principle of least work, the partial derivative of strain energy with respect to B_y must be zero; that is,

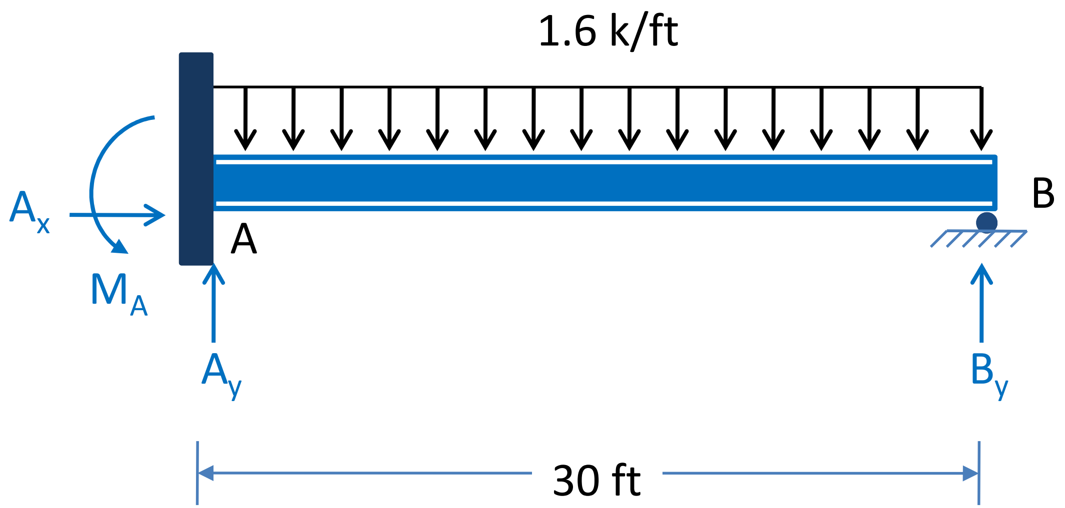
$$\frac{\partial U}{\partial B_y} = \int_0^L \frac{\partial M}{\partial B_y} \frac{M}{EI} dx = 0 \qquad (2)$$

Example 1

Determine the reactions for the beam shown in Fig., by the method of least work. *EI* is constant.

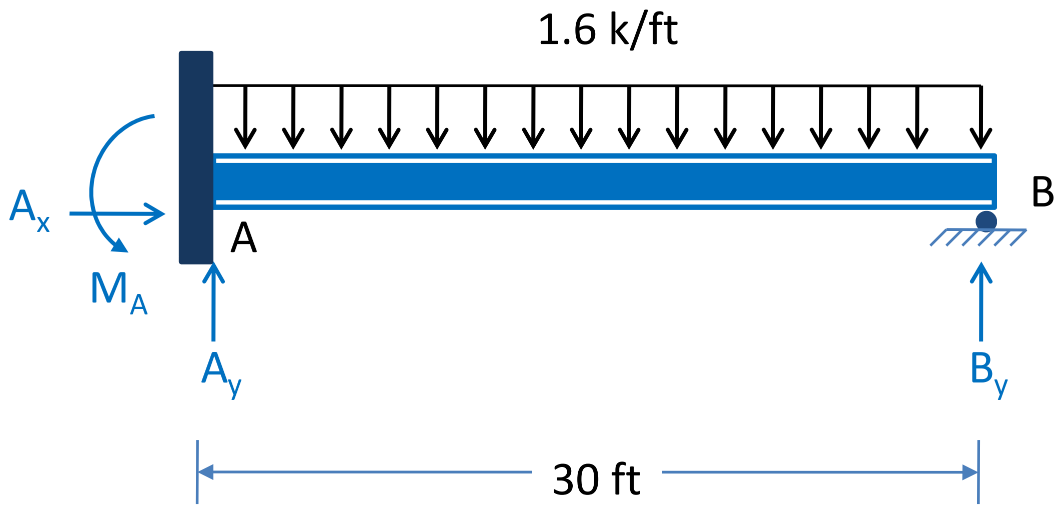


Solution



- The beam is supported by four reactions, so its degree of indeterminacy is equal to *1*.
- The vertical reaction B_y, at the roller support *B*, is selected as the redundant.

Solution



- We will evaluate the magnitude of the redundant by minimizing the strain energy of the beam with respect to B_y.
- The strain energy of a beam subjected only to bending can be expressed as

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad (1)$$

Solution

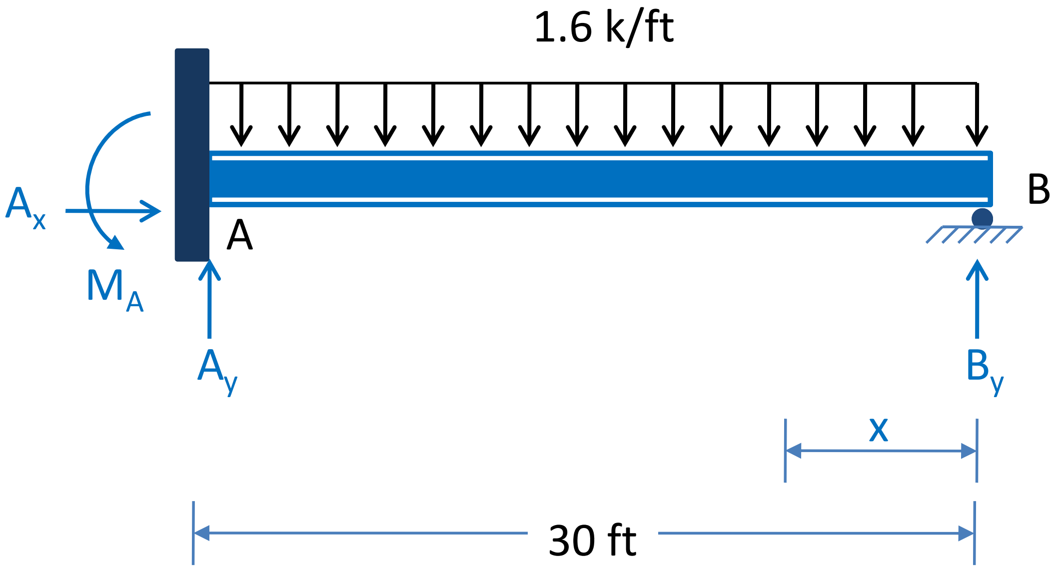
$$U = \int_0^L \frac{M^2}{2EI} dx \qquad (1)$$

- According to the principle of least work, the partial derivative of strain energy with respect to B_y must be zero; that is,

$$\frac{\partial U}{\partial B_y} = \int_0^L \frac{\partial M}{\partial B_y} \frac{M}{EI} dx = 0 \qquad (2)$$

- Using the x coordinate shown in Fig, we write the equation for bending moment, *M*, in terms of B_y, as

Solution



$$M = B_y(x) - \frac{1.6x^2}{2}$$

- Next, we partially differentiate the expression for *M* w.r.t B_y to obtain

$$\frac{\partial M}{\partial B_y} = x$$

Solution

- By substituting the expression for *M* and $\partial M/\partial B_y$ into Eq. (2), we write

$$\frac{\partial U}{\partial B_y} = \int_0^L \frac{\partial M}{\partial B_y}\frac{M}{EI}dx = 0 \qquad (2)$$

$$\frac{\partial M}{\partial B_y} = x \qquad M = B_y(x) - \frac{1.6x^2}{2},$$

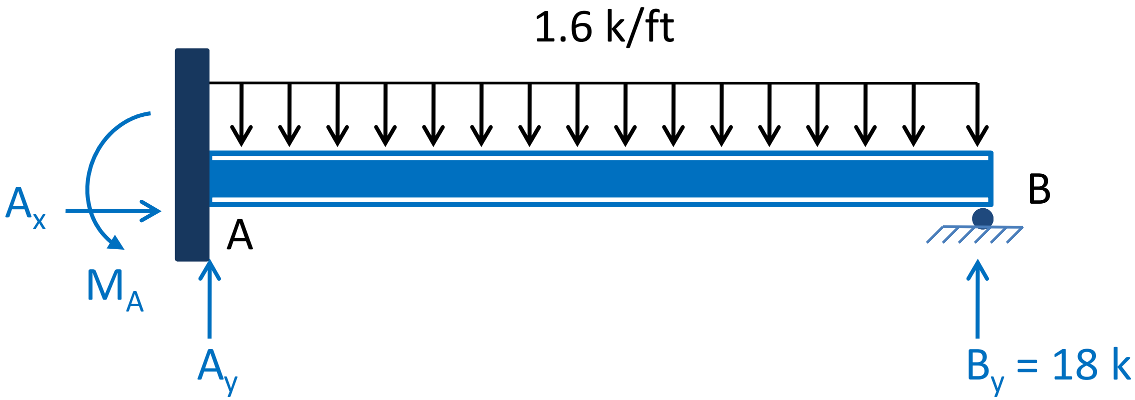
$$\frac{1}{EI}\left[\int_0^{30} x\left(B_y x - 0.8x^2\right)dx\right] = 0$$

By integrating we, obtain

$$9{,}000B_y - 162{,}000 = 0$$

$$B_y = 18\,k \uparrow$$

Solution



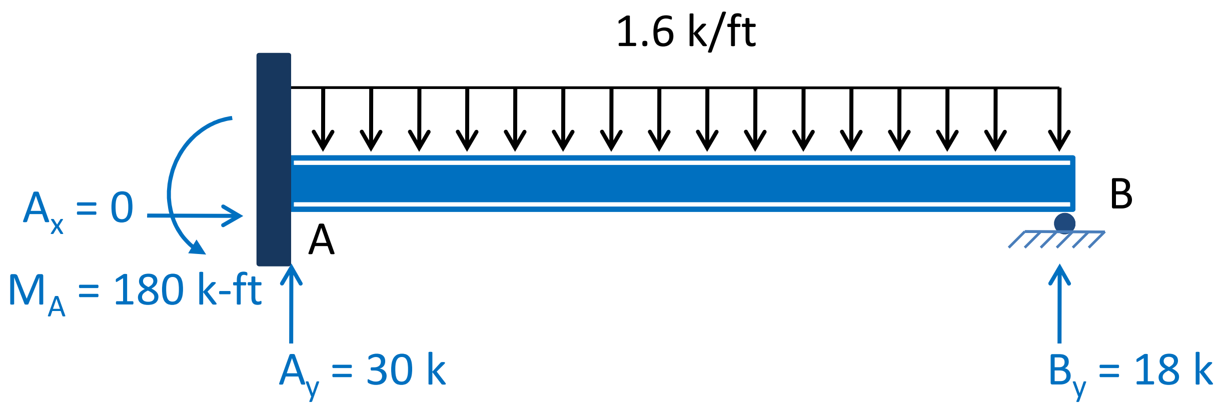
- To determine the remaining reactions of the indeterminate beam, we apply the equations of equilibrium

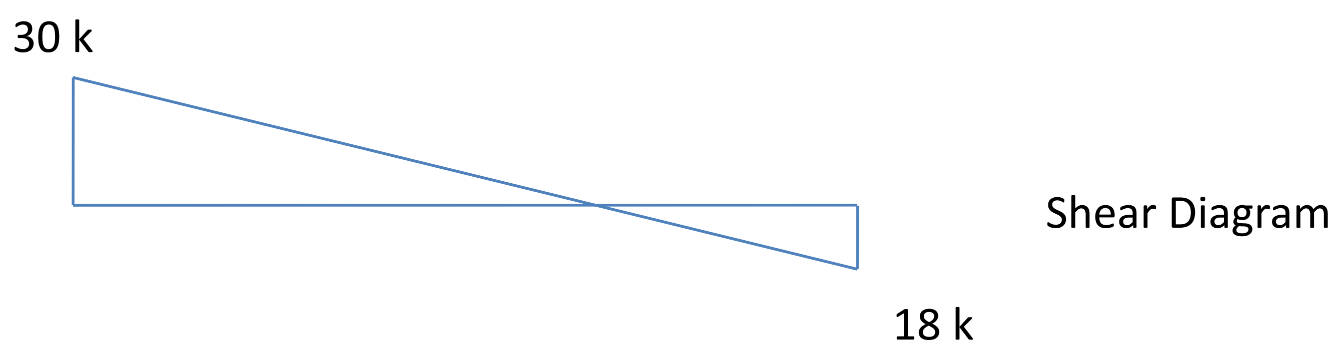
$$+\rightarrow \sum F_x = 0 \qquad\qquad A_x = 0 \quad \text{ANS}$$

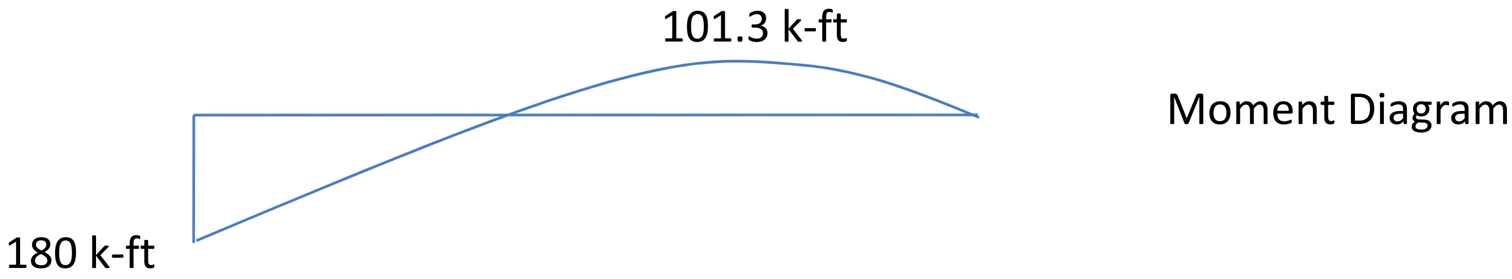
$$+\uparrow \sum F_y = 0 \qquad A_y - 1.6(30) + 18 = 0 \qquad A_y = 30\,k \uparrow \quad \text{ANS}$$

$$+\circlearrowleft \sum M_A = 0 \qquad M_A - 1.6(30)(15) + 18(30) = 0 \qquad M_A = 180\,k - ft \circlearrowright \text{ANS}$$

Solution

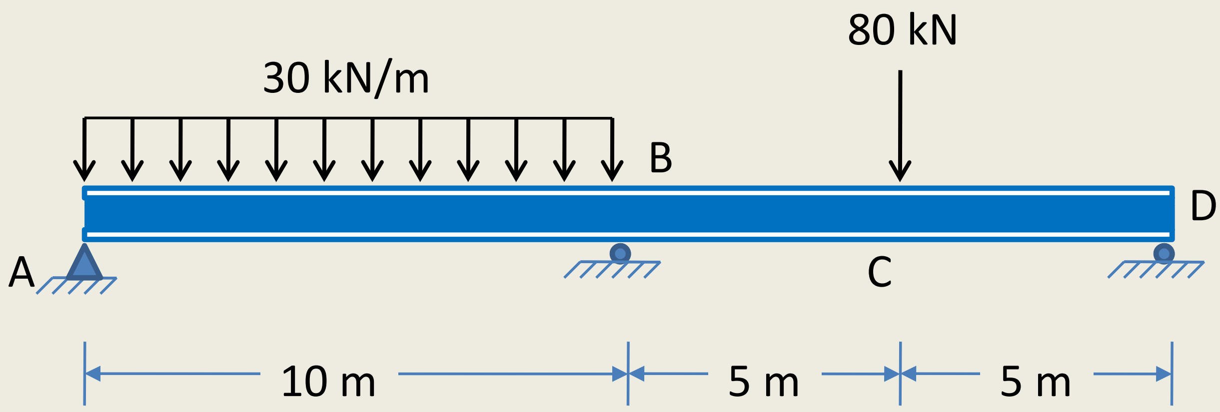

1.6 k/ft
A_x = 0
M_A = 180 k-ft
A
B
A_y = 30 k
B_y = 18 k


30 k
18 k
Shear Diagram

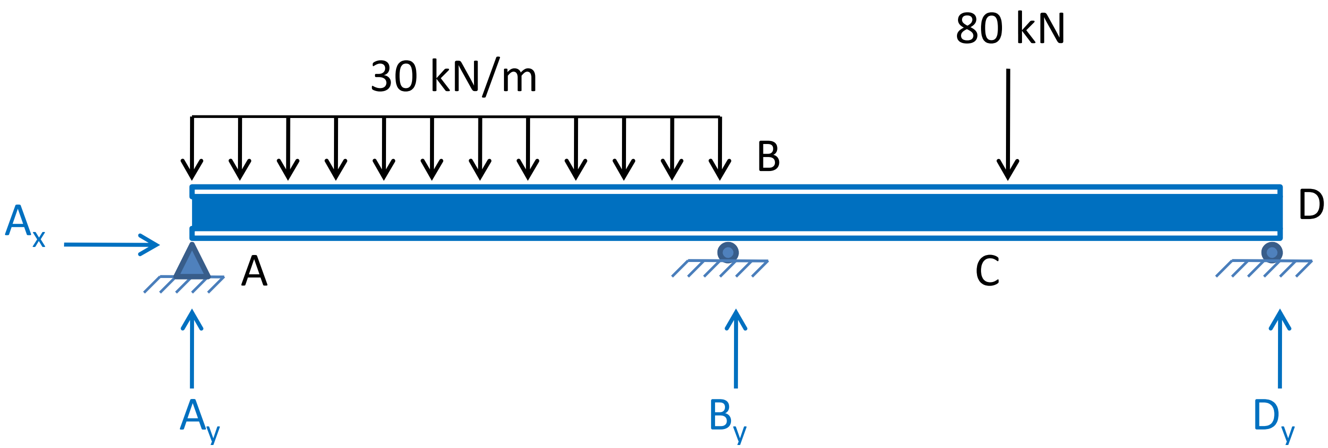

101.3 k-ft
180 k-ft
Moment Diagram

Example 2

Determine the reactions for the two-span continuous beam shown in Fig., by the method of least work. *EI* is constant.



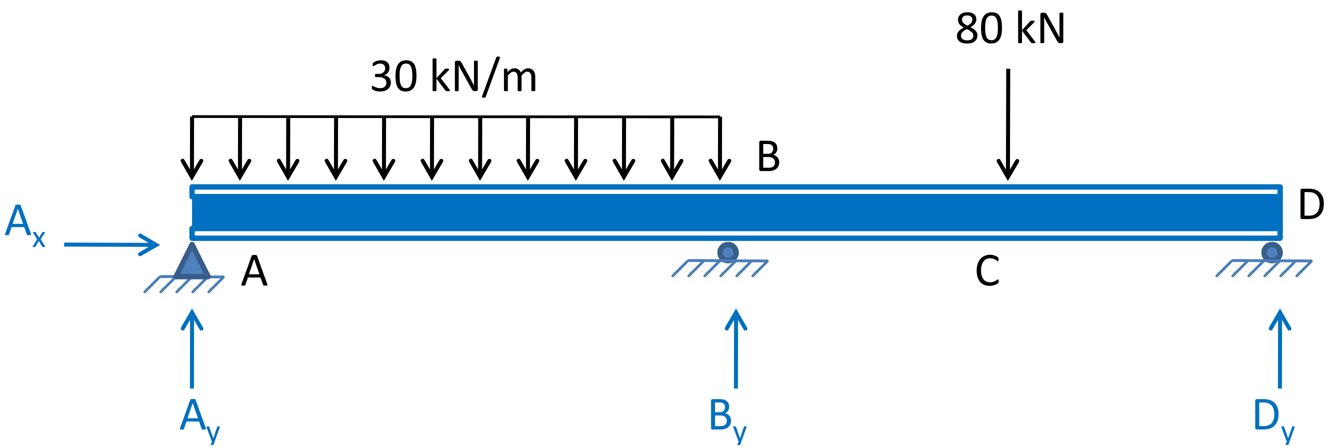
Solution



- The beam is supported by four reactions. Since there are only three equilibrium equations, the degree of indeterminacy of the beam is equal to 1.

- Let us select the reaction B_y to be the redundant.

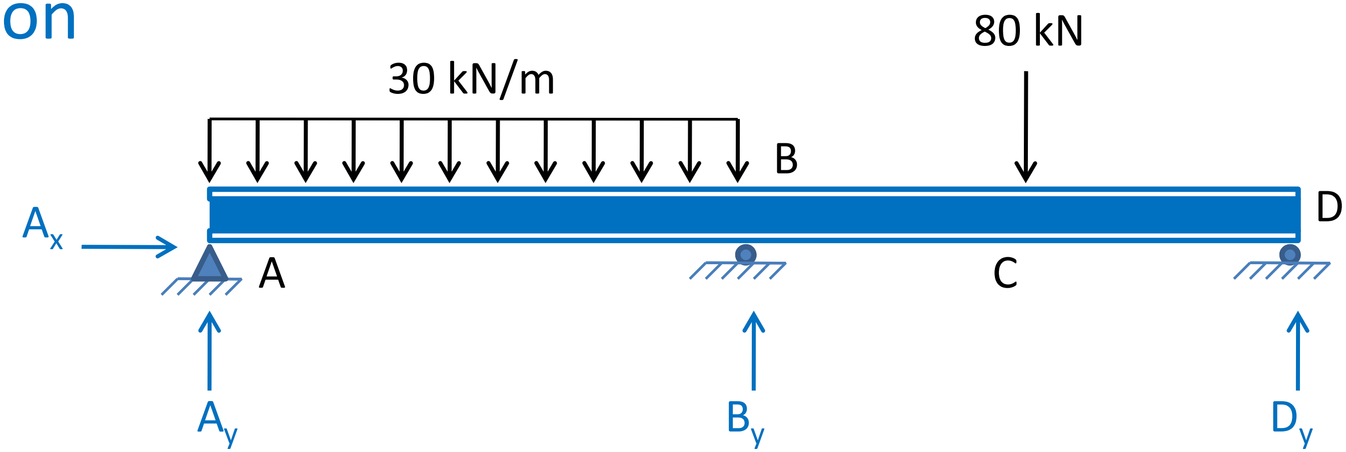
Solution



- The magnitude of the redundant will be determined by minimizing the strain energy of the beam with respect to B_y.
- The strain energy of a beam subjected only to bending is

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad (1)$$

Solution

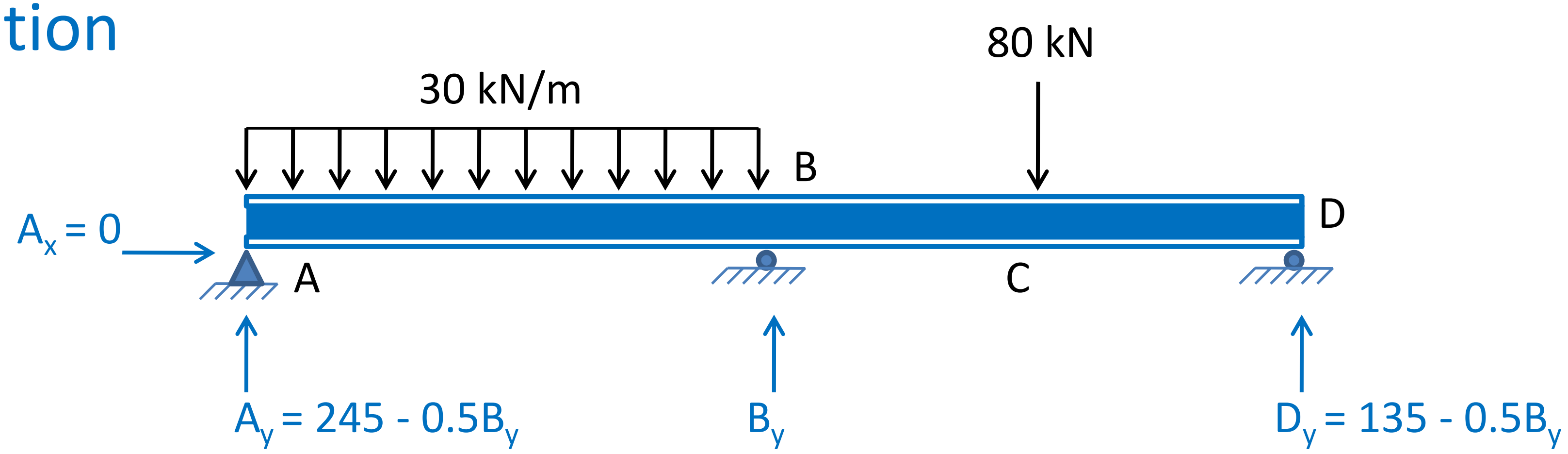


- According to the Principle of Least Work.

$$\frac{\partial U}{\partial B_y} = \int_0^L \frac{\partial M}{\partial B_y} \frac{M}{EI} dx = 0 \qquad (2)$$

- Before we can obtain the equations for bending moments, M, we must express the reactions at the supports A and D of the beam in terms of the redundant B_y.

Solution



- Applying the three equilibrium equations, we write

$$+\rightarrow \sum F_x = 0 \qquad A_x = 0 \qquad \text{ANS}$$

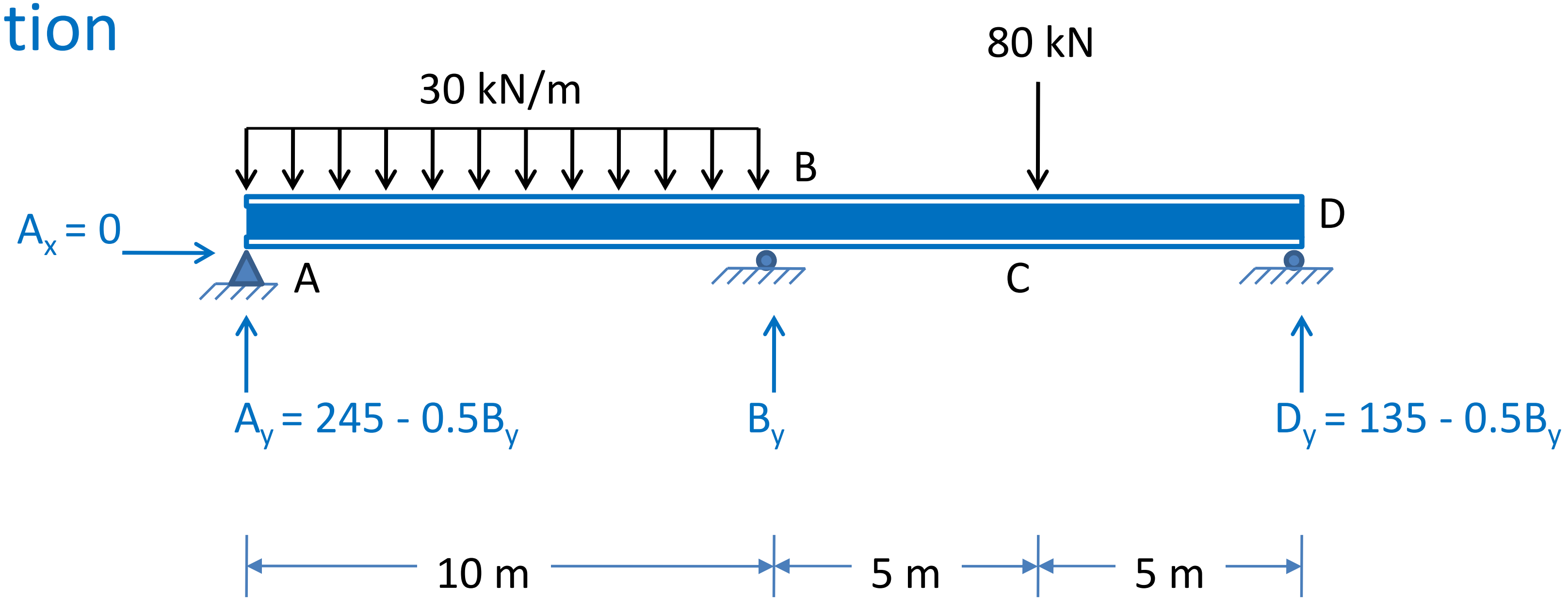
$$+\circlearrowleft \sum M_D = 0$$

$$-A_y(20) + 30(10)(15) - B_y(10) + 80(5) = 0 \qquad A_y = 245 - 0.5B_y \quad (3)$$

$$+\uparrow \sum F_y = 0$$

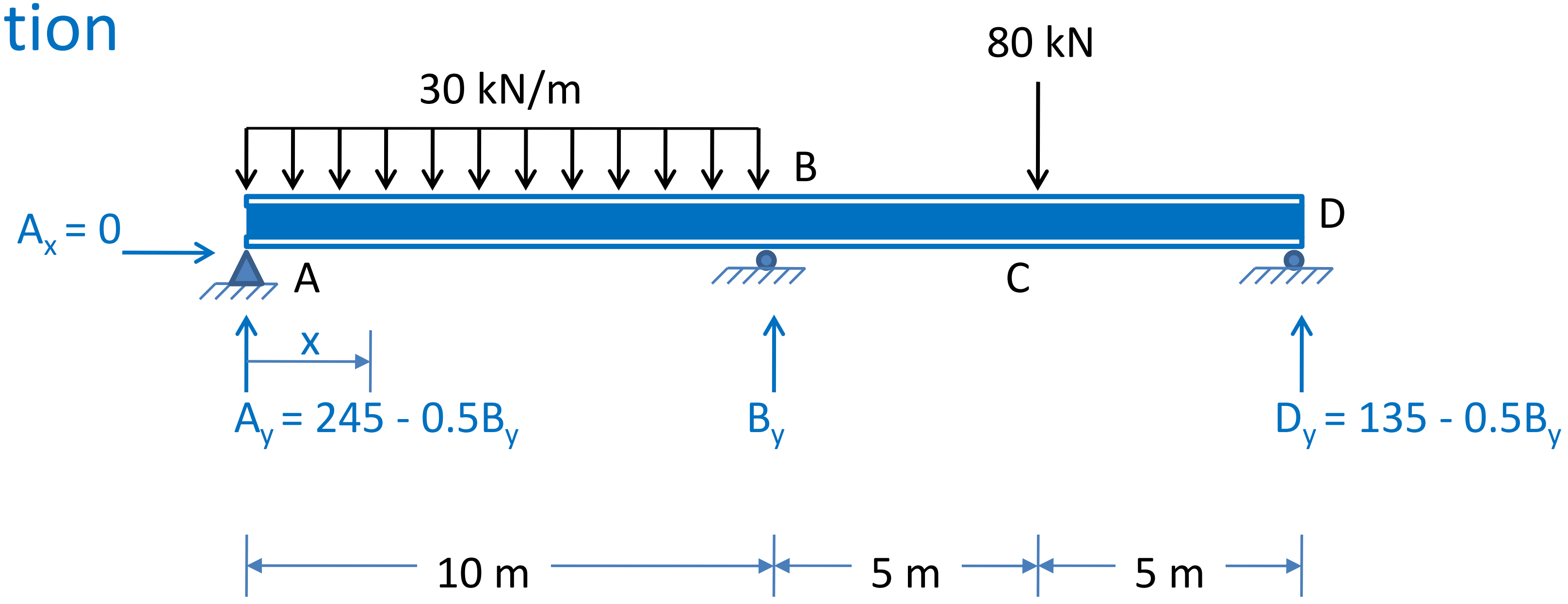
$$(245 - 0.5B_y) - 30(10) + B_y - 80 + D_y = 0 \qquad D_y = 135 - 0.5B_y \quad (4)$$

Solution



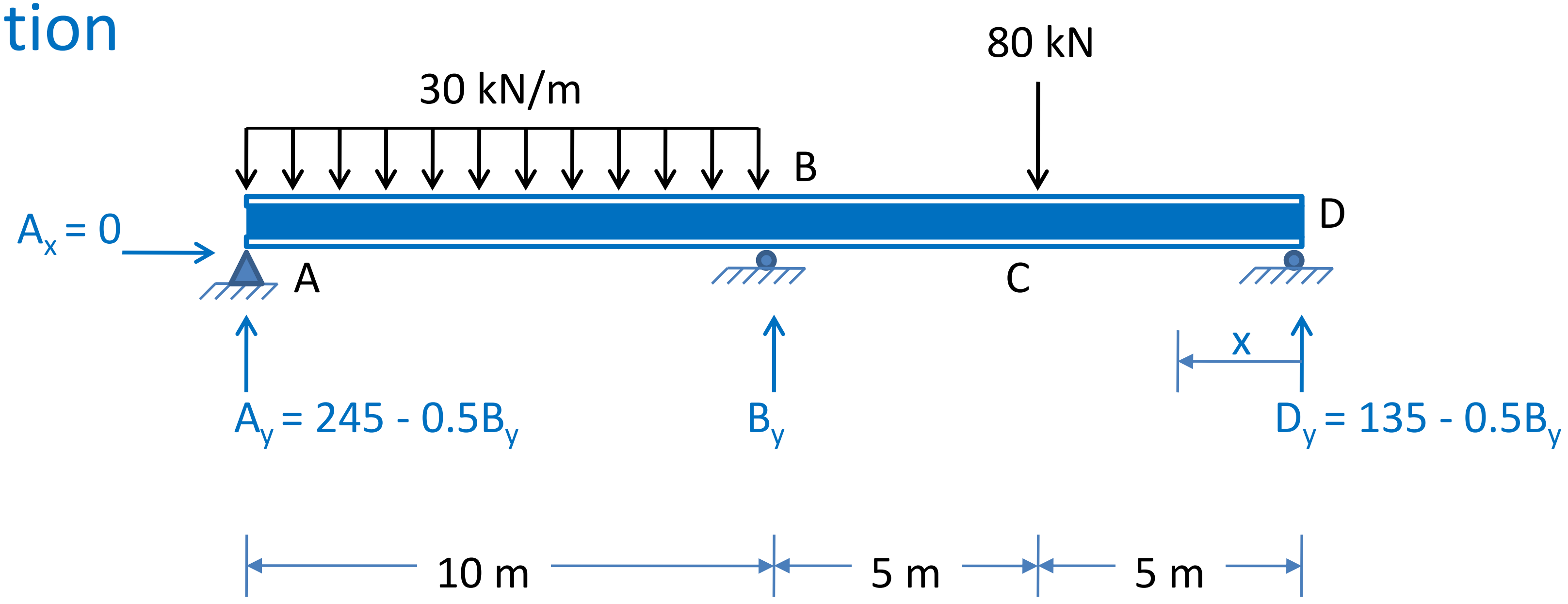
- To determine the equations for bending moments, M, the beam is divided into three segments, AB, BC, and CD.
- The x coordinates used for determining the equations are shown in Figure.
- The bending moment equations , in terms of B_y, are tabulated in Table on next slide.

Solution



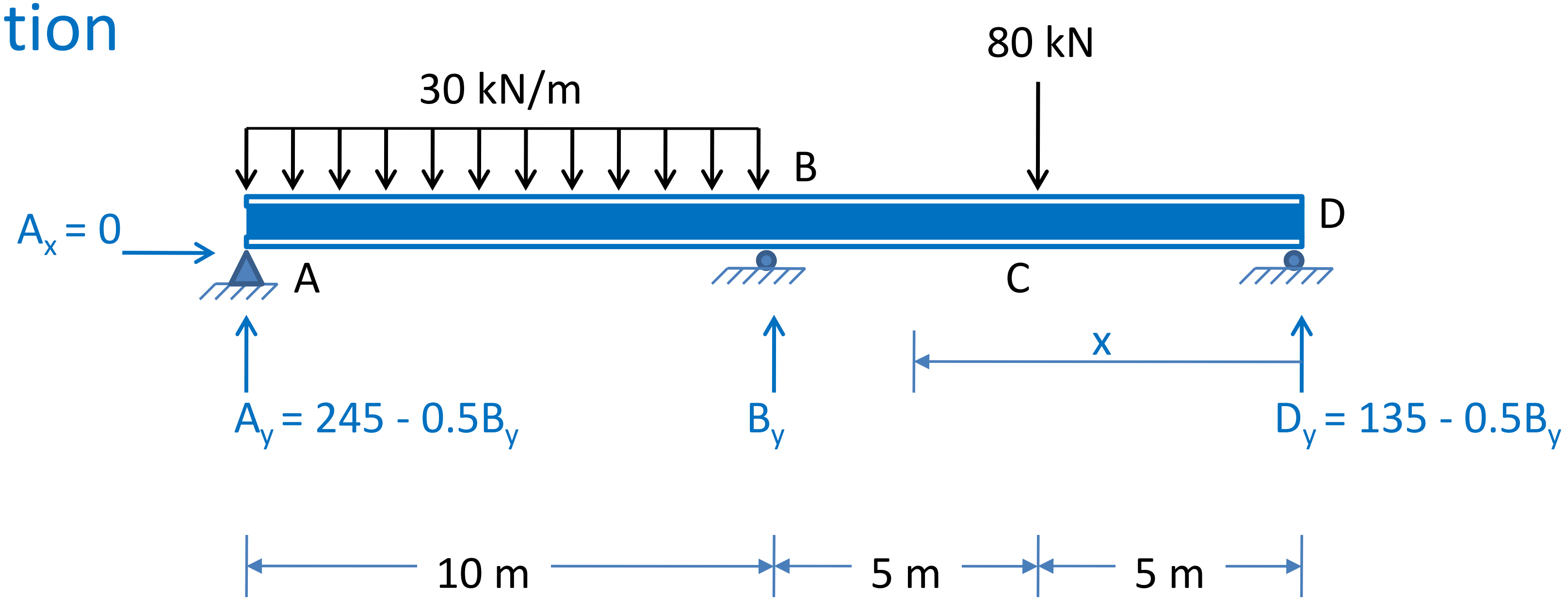
Segment	Origin	Limits	*M*	$\partial M/\partial B_y$
AB	A	0 – 10	$(245 - 0.5B_y)x - 15x^2$	-0.5x

Solution



Segment	Origin	Limits	*M*	$\partial M/\partial B_y$
AB	A	0 – 10	$(245 - 0.5B_y)x - 15x^2$	-0.5x
DC	D	0 – 5	$(135 - 0.5B_y)x$	-0.5x

Solution



Segment	Origin	Limits	*M*	***∂M/∂B_y***
AB	A	0 – 10	$(245 - 0.5B_y)x - 15x^2$	-0.5x
DC	D	0 – 5	$(135 - 0.5B_y)x$	-0.5x
CB	D	5 – 10	$(135 - 0.5B_y)x - 80(x - 5)$	-0.5x

Solution

- By substituting the expressions for M and $\partial M/\partial B_y$ into Eq. (2), we write

$$\frac{1}{EI}\int_0^{10}(-0.5x)\left(245x-0.5B_yx-15x^2\right)dx+$$
$$\frac{1}{EI}\int_0^{5}(-0.5x)\left(135x-0.5B_yx\right)dx+$$
$$\frac{1}{EI}\int_5^{10}(-0.5x)\left(55x-0.5B_yx+400\right)dx=0$$

- By integrating, we obtain

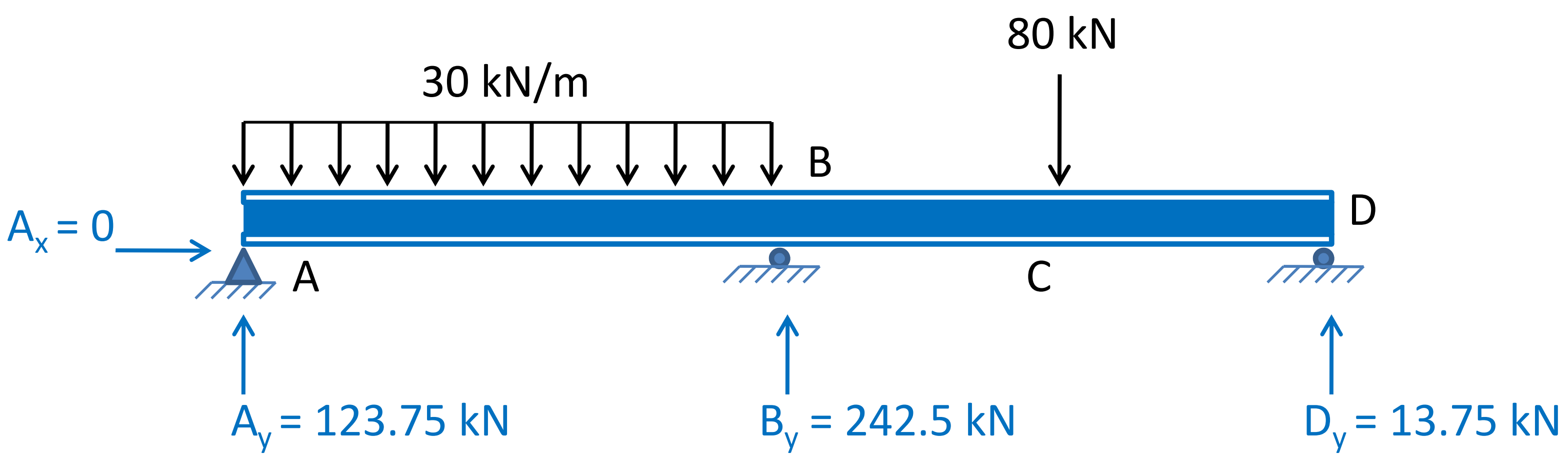
$$-40{,}416.667+166.667B_y=0 \qquad \Rightarrow \qquad B_y=242.5\ kN\uparrow \qquad \text{ANS}$$

Solution

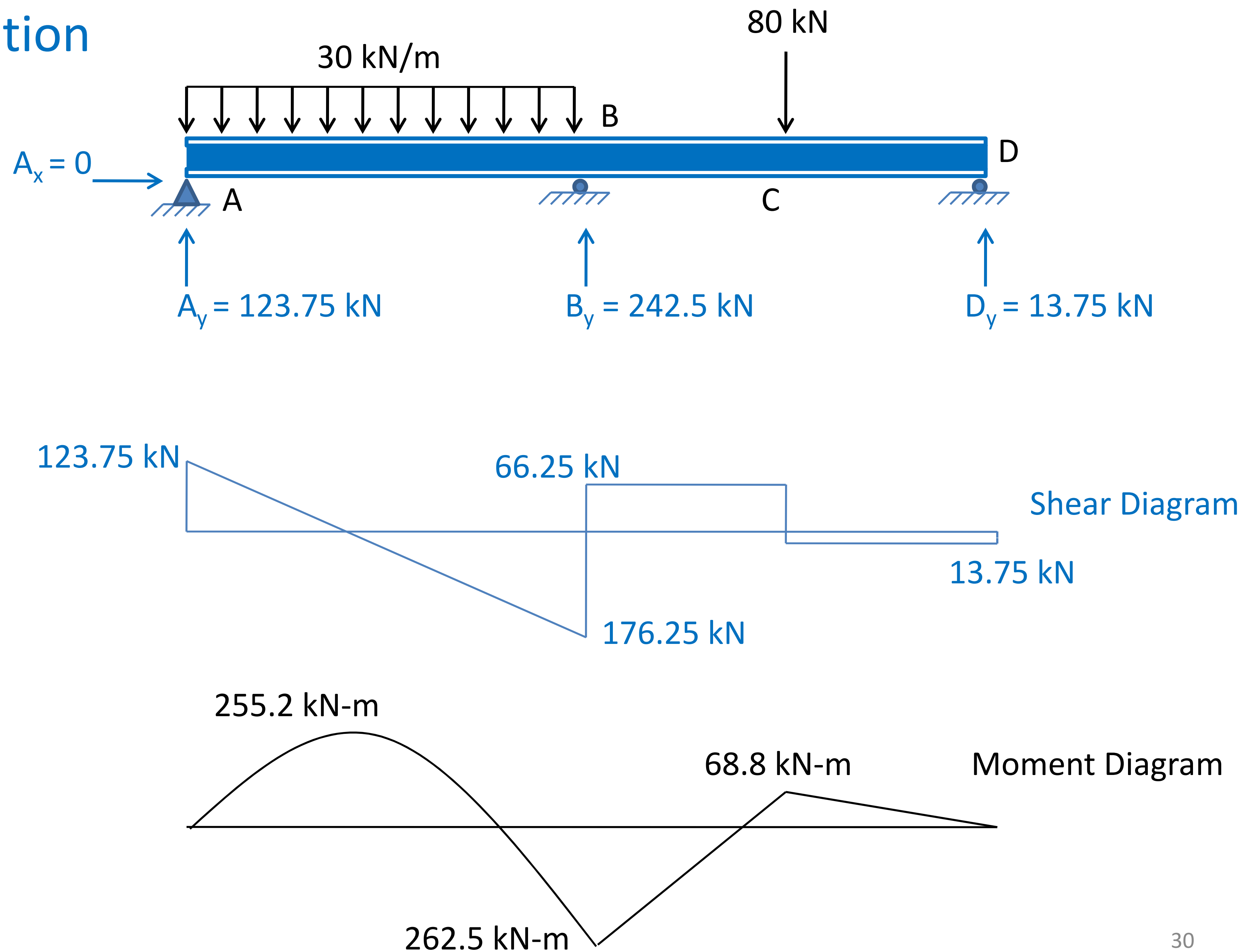
- By substituting the value of B_y into Eqs. (3) and (4), respectively, we determine the vertical reactions at supports A and D.

$$A_y = 123.75\ kN \uparrow \qquad \text{ANS}$$

$$D_y = 13.75\ kN \uparrow \qquad \text{ANS}$$



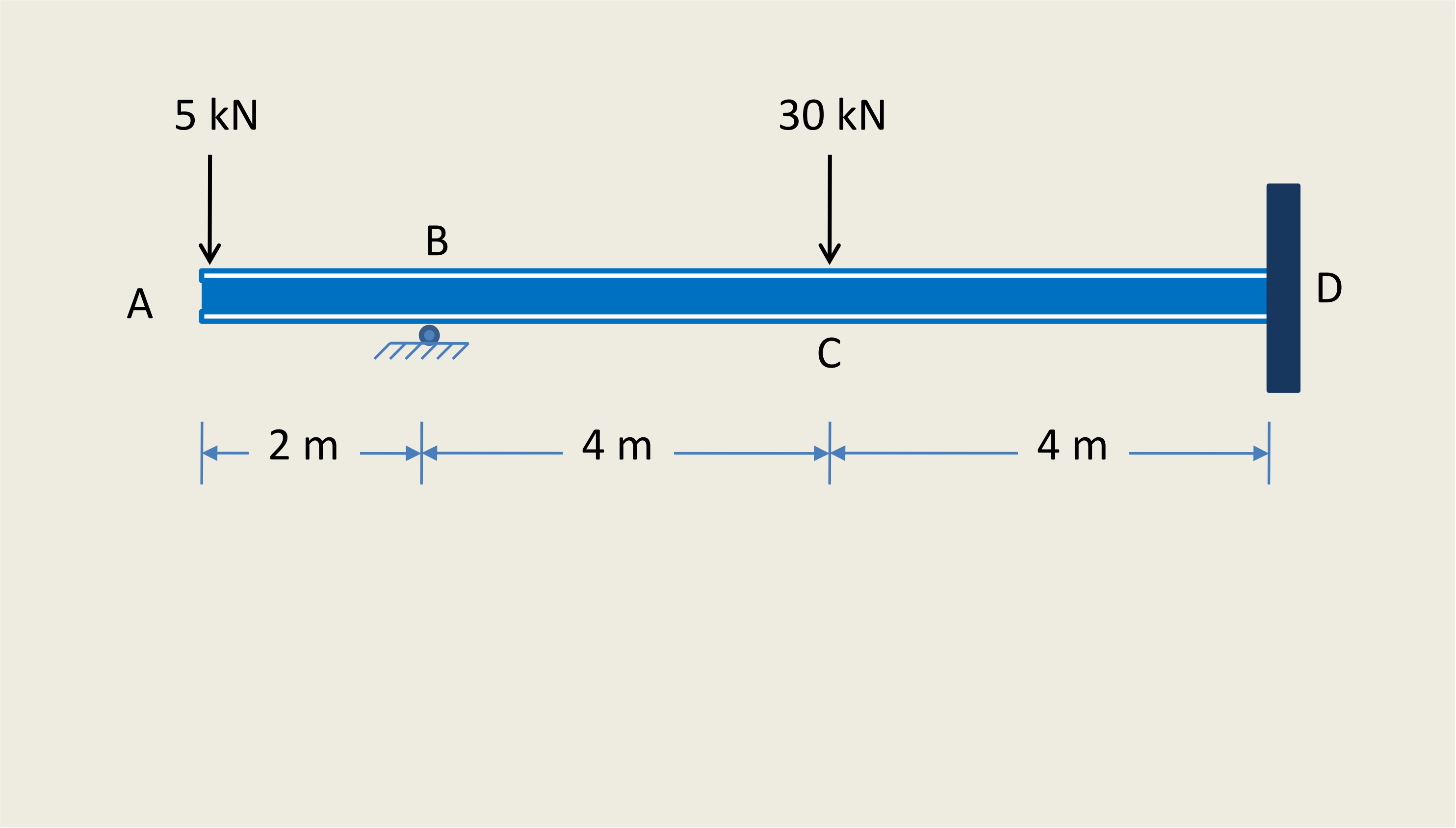
Solution



30 kN/m
80 kN
B
D
$A_x = 0$
A
C
$A_y = 123.75$ kN
$B_y = 242.5$ kN
$D_y = 13.75$ kN
123.75 kN
66.25 kN
Shear Diagram
13.75 kN
176.25 kN
255.2 kN-m
68.8 kN-m
Moment Diagram
262.5 kN-m

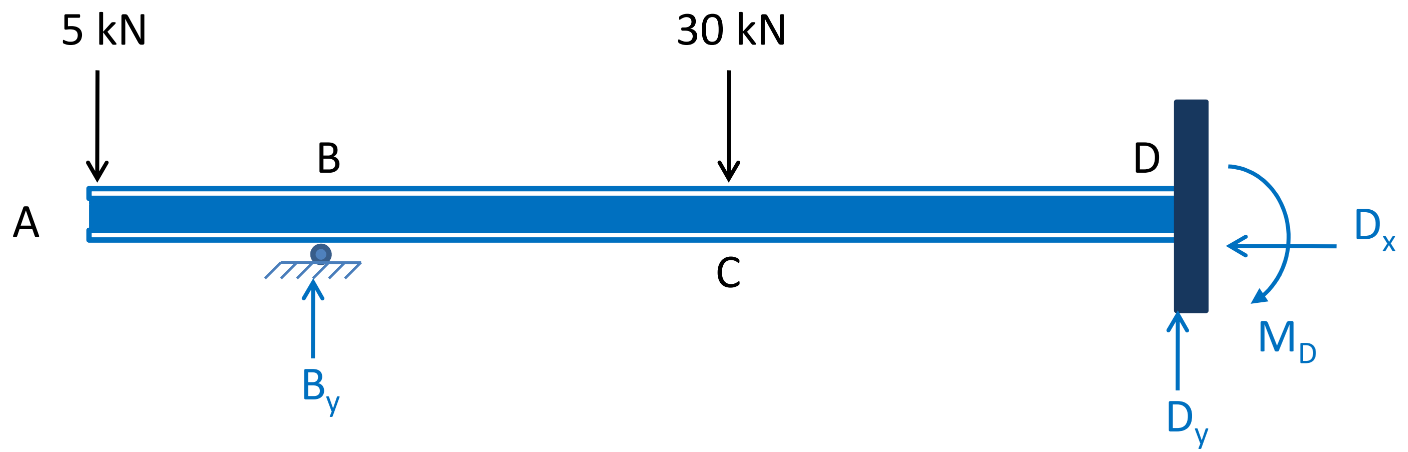
Example 3

Determine the reactions for the beam shown in Fig., by the method of least work. *EI* is constant.



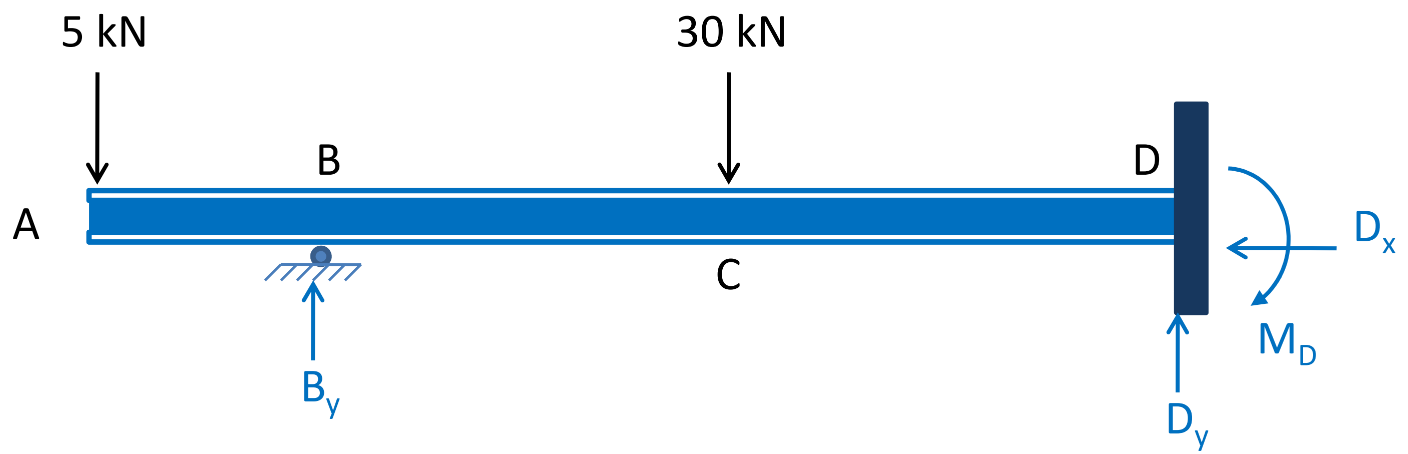
Solution

The beam is supported by four reactions. The equations of equilibrium is three, so the beam is indeterminate to the first degree.



- Let us select the reaction B_y to be the redundant.

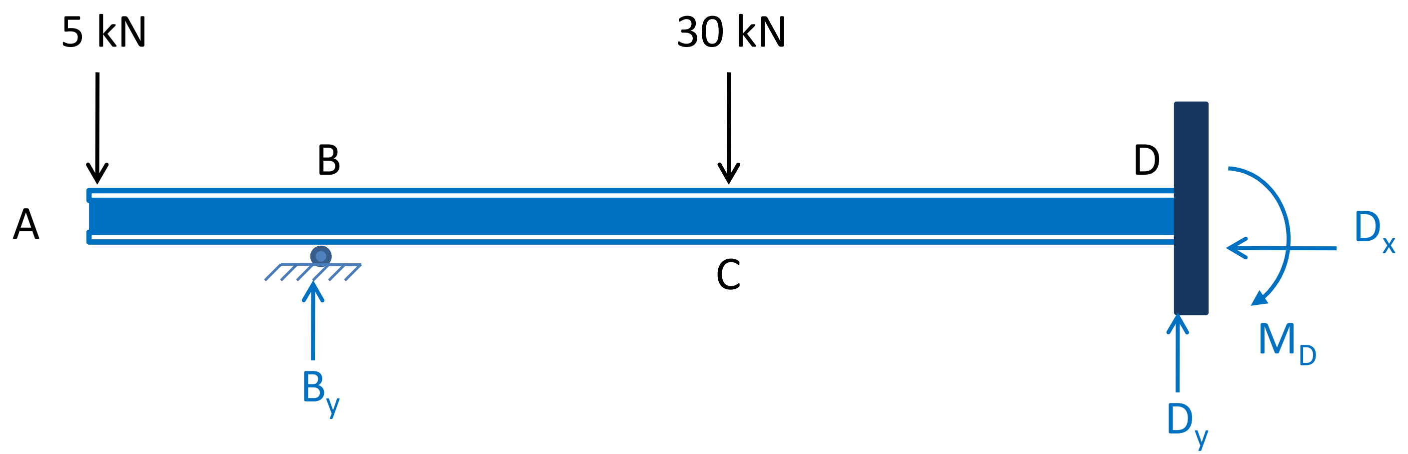
Solution



- The magnitude of the redundant will be determined by minimizing the strain energy of the beam with respect to B_y.
- The strain energy of a beam subjected only to bending is

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad (1)$$

Solution



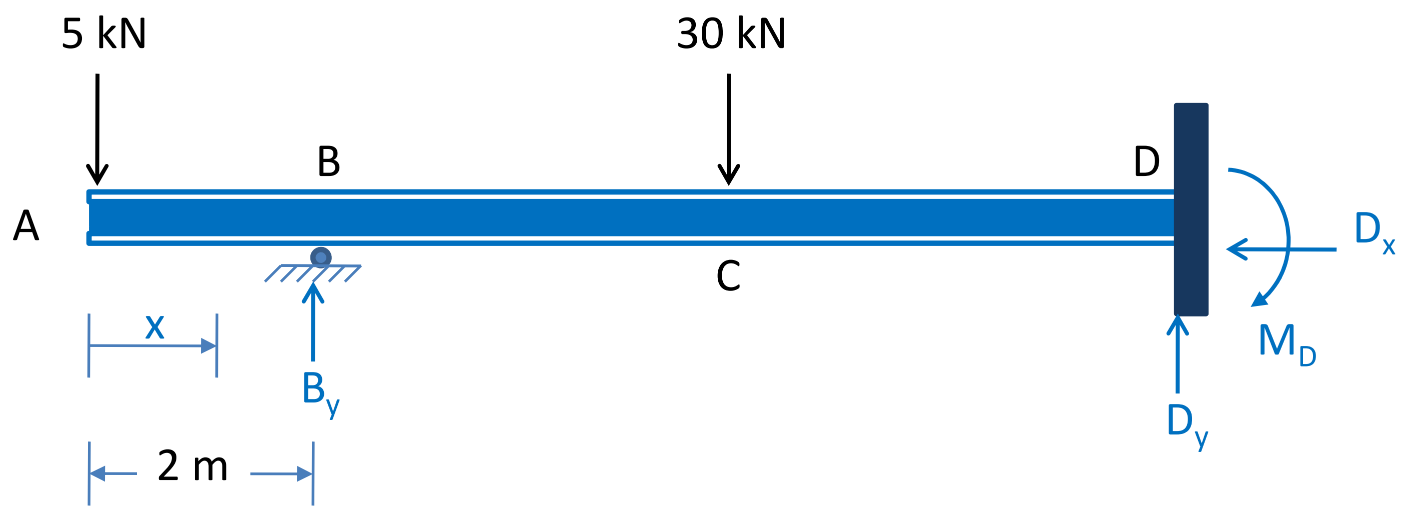
- According to the Principle of Least Work.

$$\frac{\partial U}{\partial B_y} = \int_0^L \frac{\partial M}{\partial B_y} \frac{M}{EI} dx = 0 \qquad (2)$$

- To determine the equations for bending moments, M, the beam is divided into three segments, AB, BC, and CD.
- The x coordinates used for determining the equations are shown in Figure on next slide.

Solution

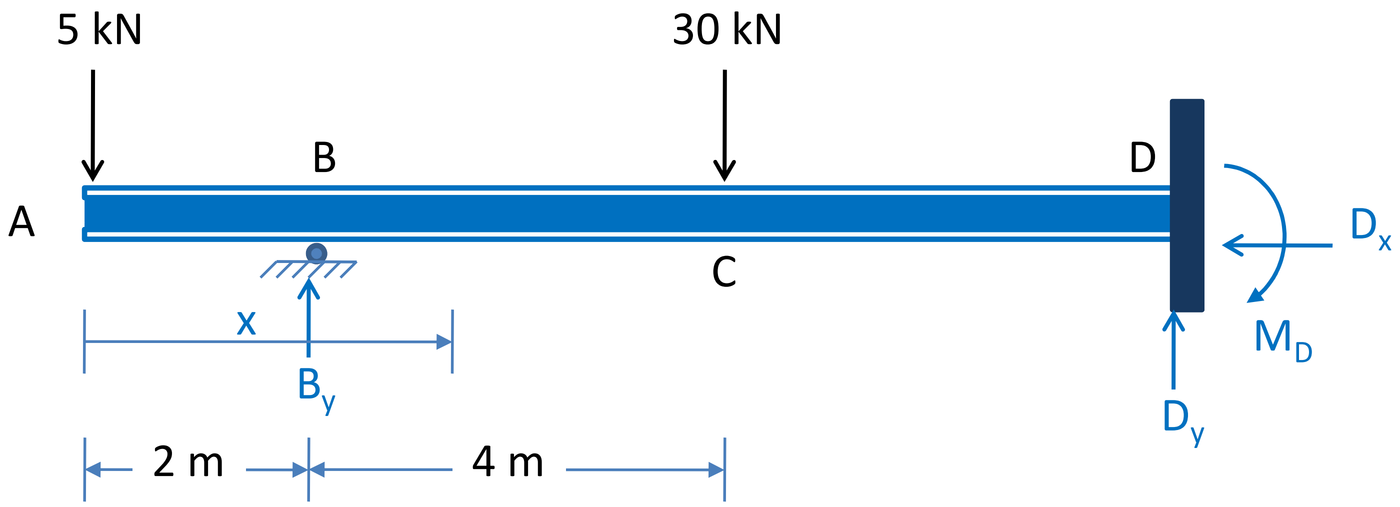
- The bending moment equations , in terms of B_y, are tabulated in Table.



Segment	Origin	Limits	*M*	*∂M/∂B_y*
AB	A	0 – 2	-5x	0

Solution

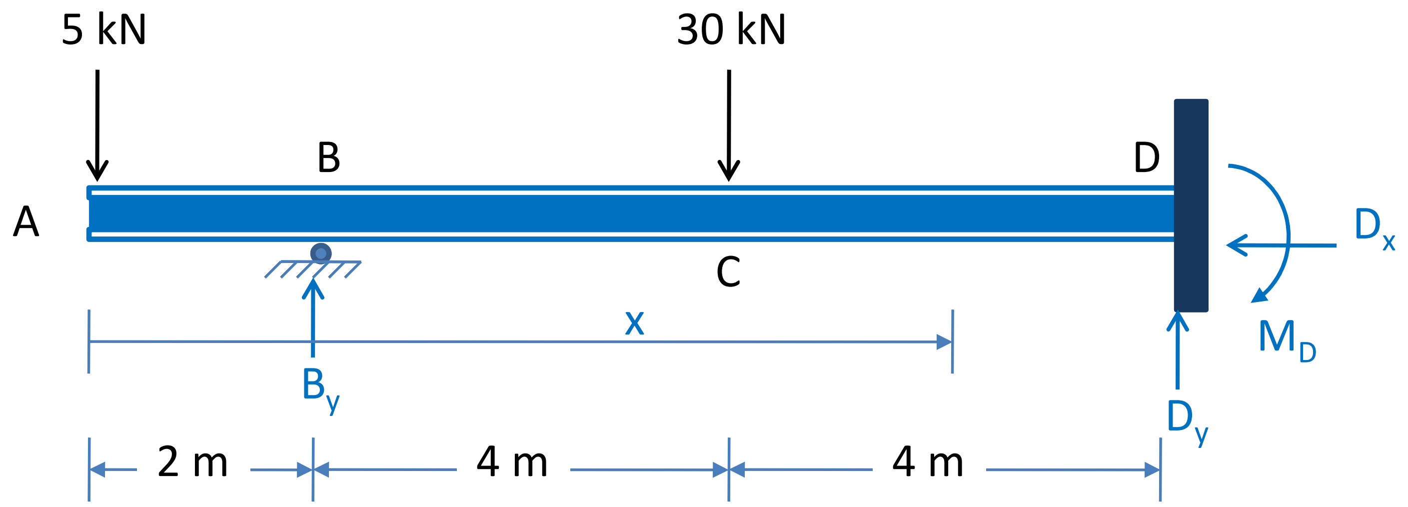
- The bending moment equations , in terms of B_y, are tabulated in Table.



Segment	Origin	Limits	*M*	**∂M/∂B**$_y$
AB	A	0 – 2	-5x	0
BC	A	2 – 6	$-5x + B_y(x-2)$	x - 2

Solution

- The bending moment equations , in terms of B_y, are tabulated in Table.



Segment	Origin	Limits	*M*	$\partial M/\partial B_y$
AB	A	0 – 2	$-5x$	0
BC	A	2 – 6	$-5x + B_y(x-2)$	$x - 2$
CD	A	6 – 10	$-5x + B_y(x-2) - 30(x-6)$	$x - 2$

Solution

- By substituting the expressions for *M* and $\partial M/\partial B_y$ into Eq. (2), we write

$$\frac{\partial U}{\partial B_y} = \int_0^L \frac{\partial M}{\partial B_y}\frac{M}{EI}dx = 0 \qquad (2)$$

$$\frac{1}{EI}\int_0^2 (-5x)(0)dx + \frac{1}{EI}\int_2^6 (-5x + B_y(x-2))(x-2))dx +$$

$$\frac{1}{EI}\int_6^{10} (-5x + B_y(x-2) - 30(x-6))(x-2))dx = 0$$

- By integrating, we obtain

$$-2773.327 + 170.66B_y = 0 \qquad \Rightarrow \qquad B_y = 16.25\ kN \uparrow \qquad \text{ANS}$$

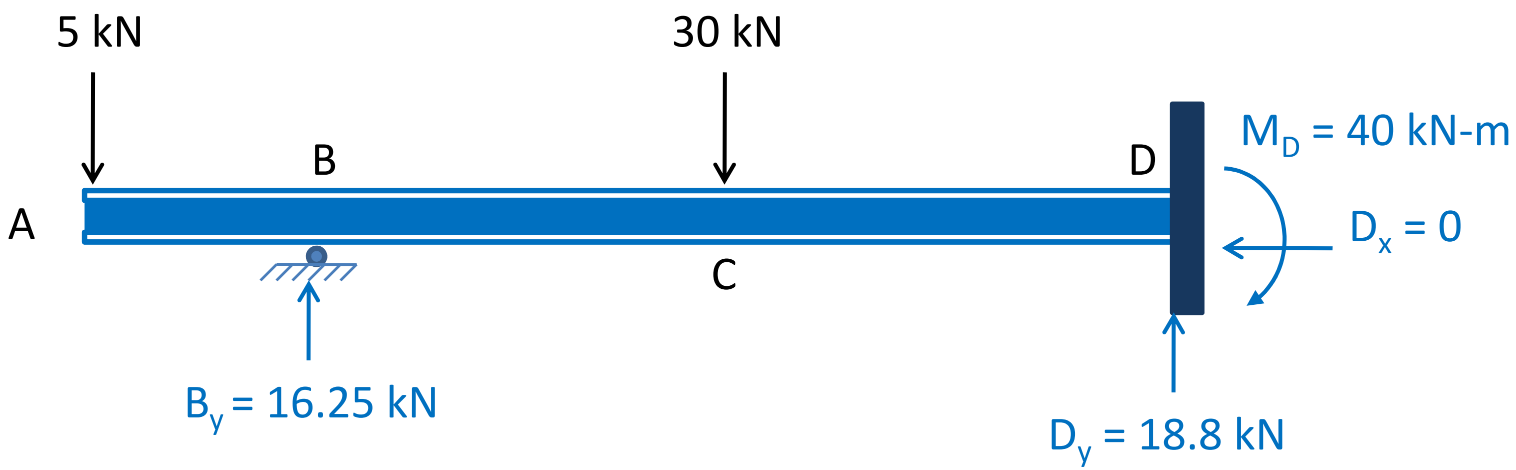
Solution

- By using the equations of equilibrium, the remaining reactions are find as

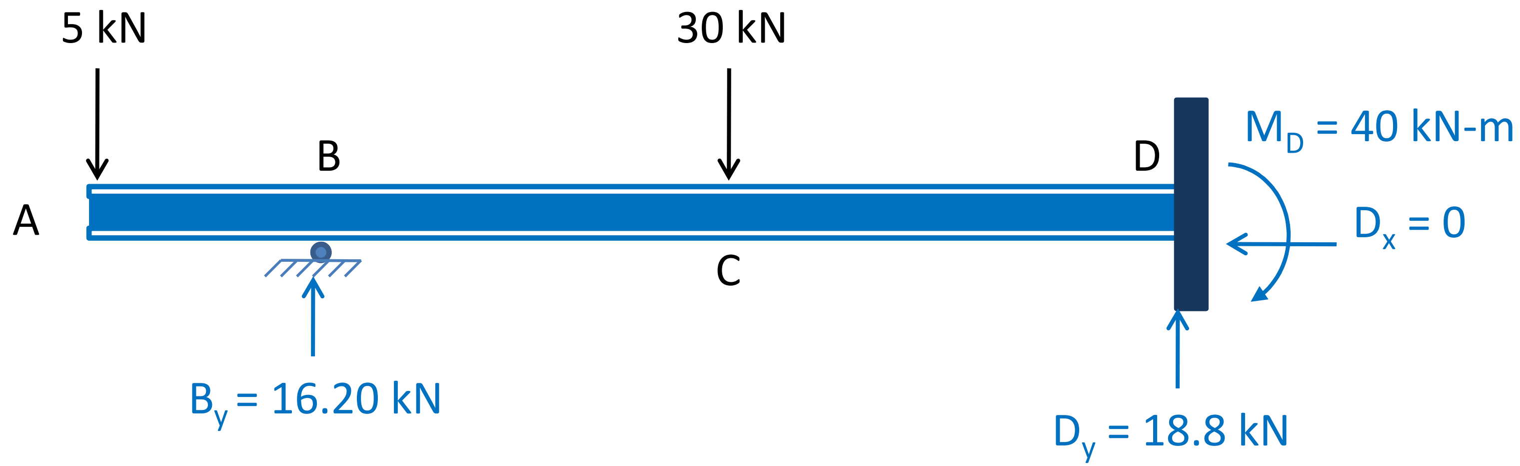
$$D_y = 18.8\ kN \uparrow \qquad \text{ANS}$$

$$D_x = 0 \qquad \text{ANS}$$

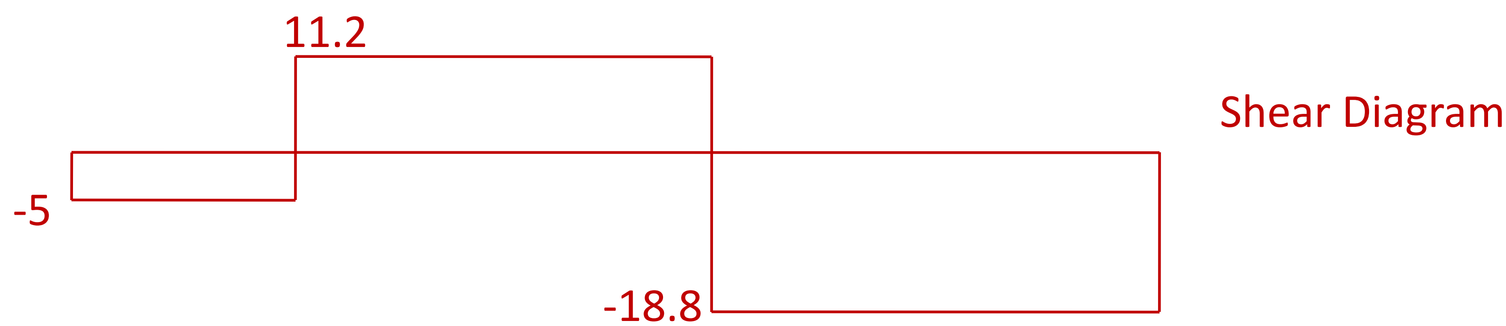
$$M_D = 40\ kN - m \curvearrowright \qquad \text{ANS}$$



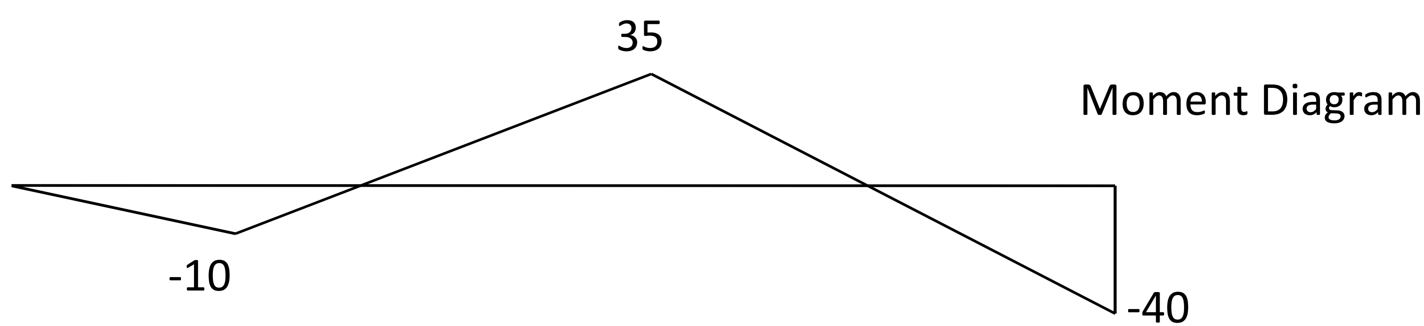
Solution



5 kN
30 kN
B
D
M_D = 40 kN-m
A
D_x = 0
C
B_y = 16.20 kN
D_y = 18.8 kN



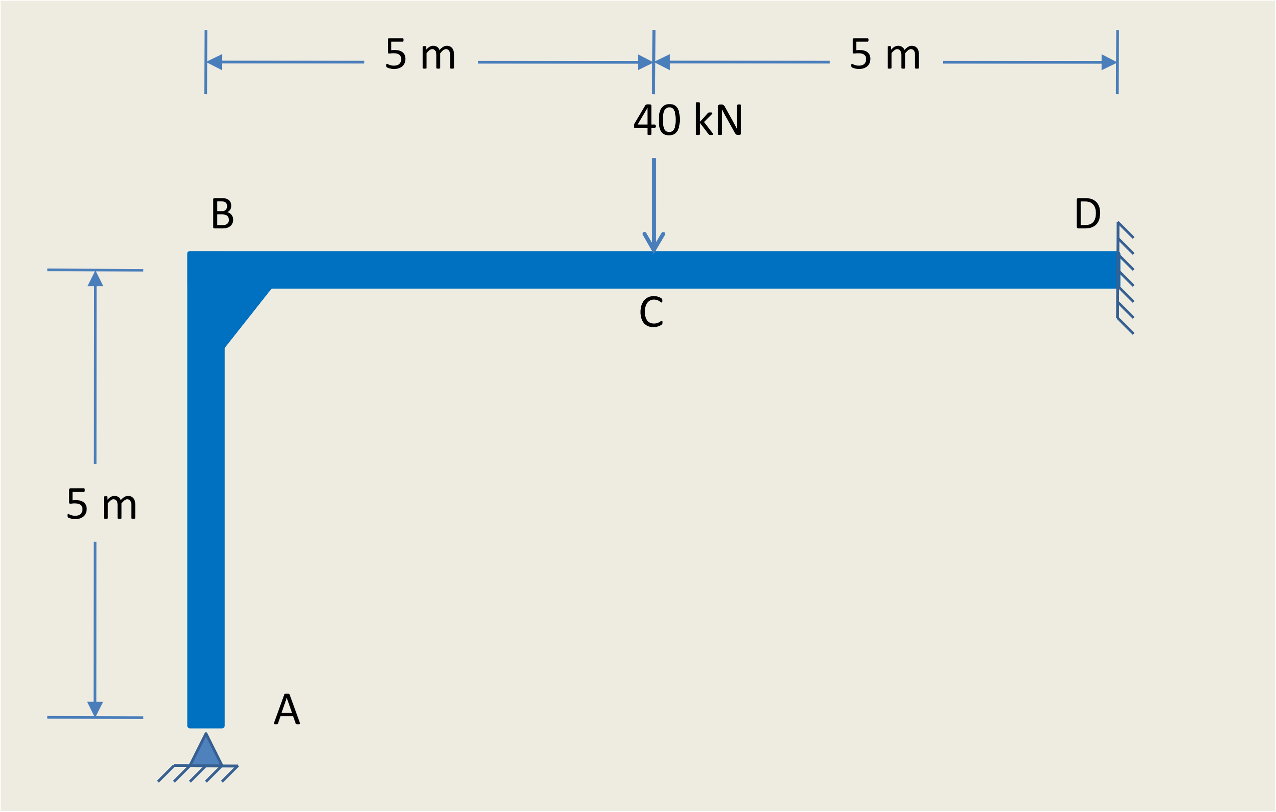
11.2
Shear Diagram
-5
-18.8



35
Moment Diagram
-10
-40

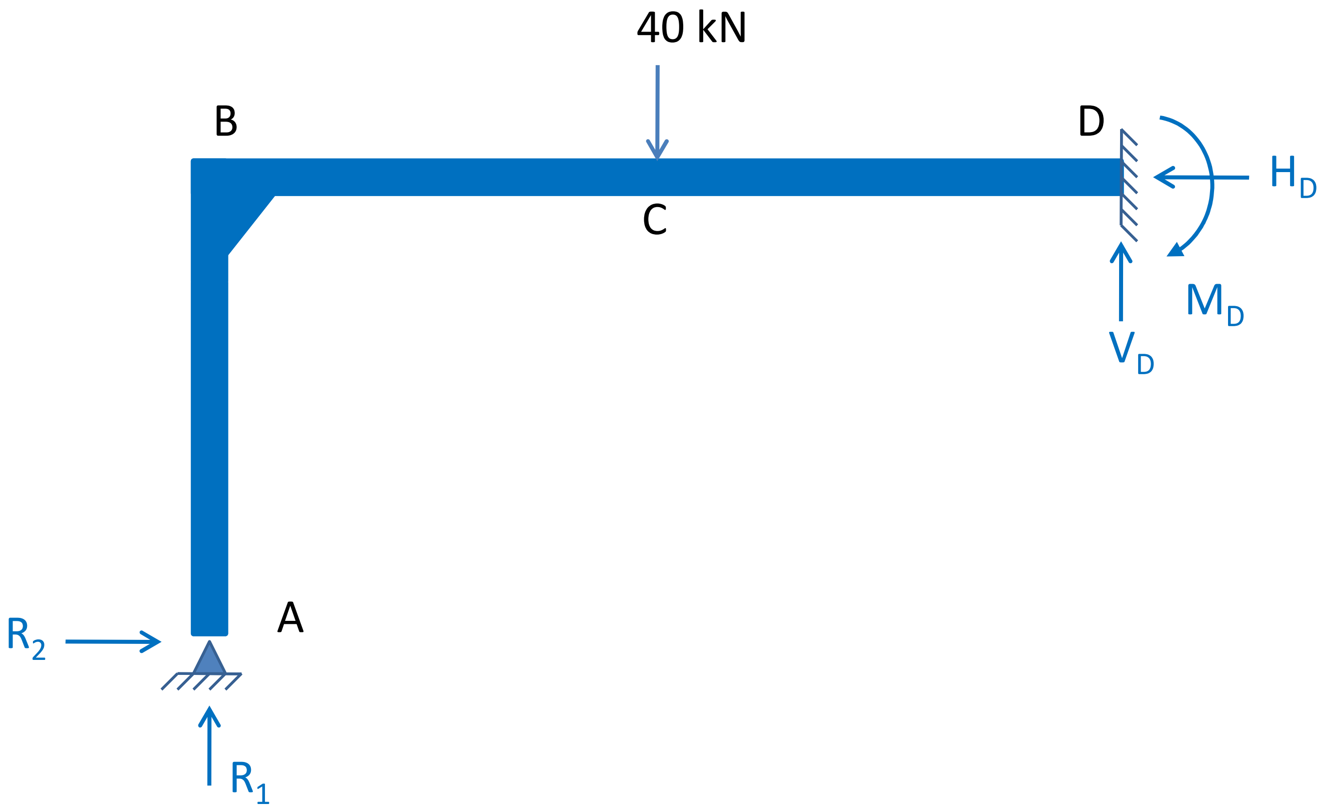
Example 4

Determine the reactions for the frame shown in Fig., by the method of least work. *EI* is constant.



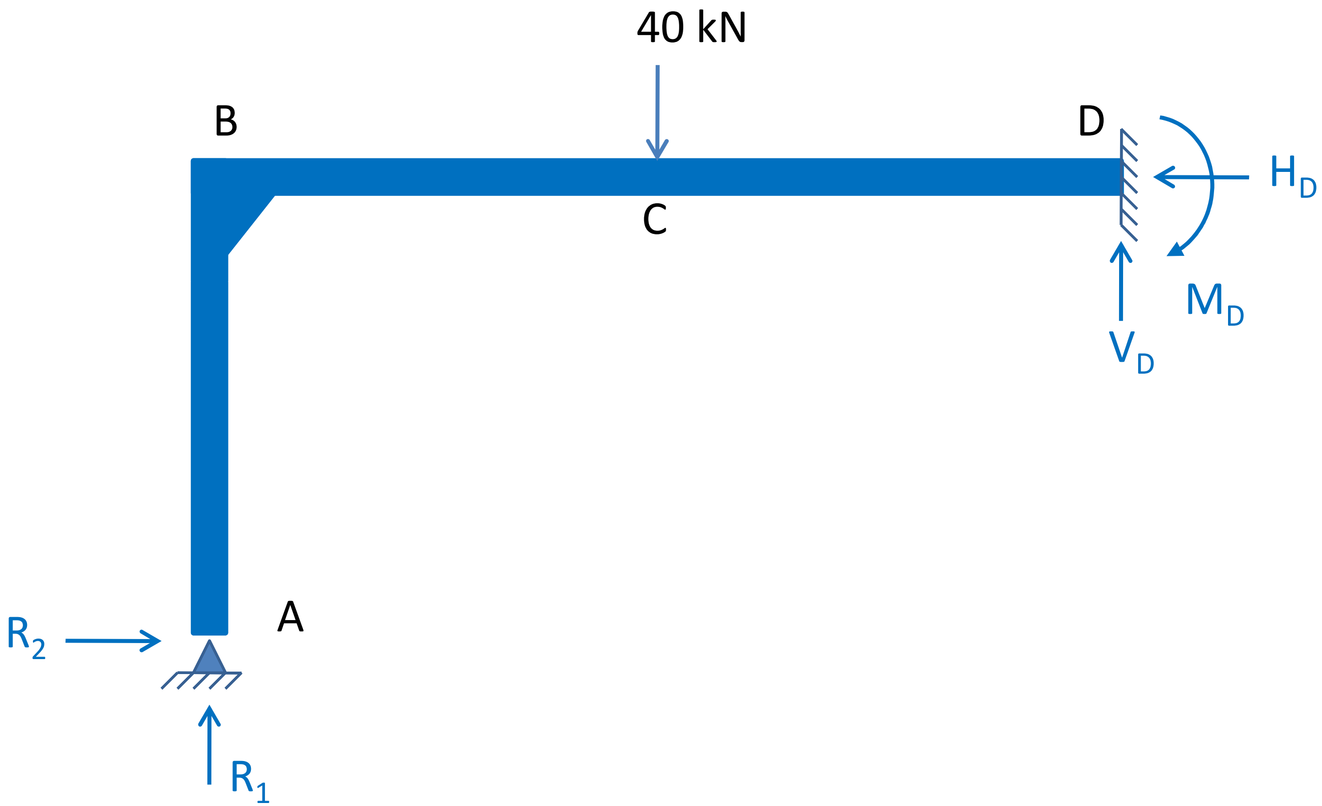
Solution

The structure is indeterminate to the 2nd degree. It has two redundant reactions.

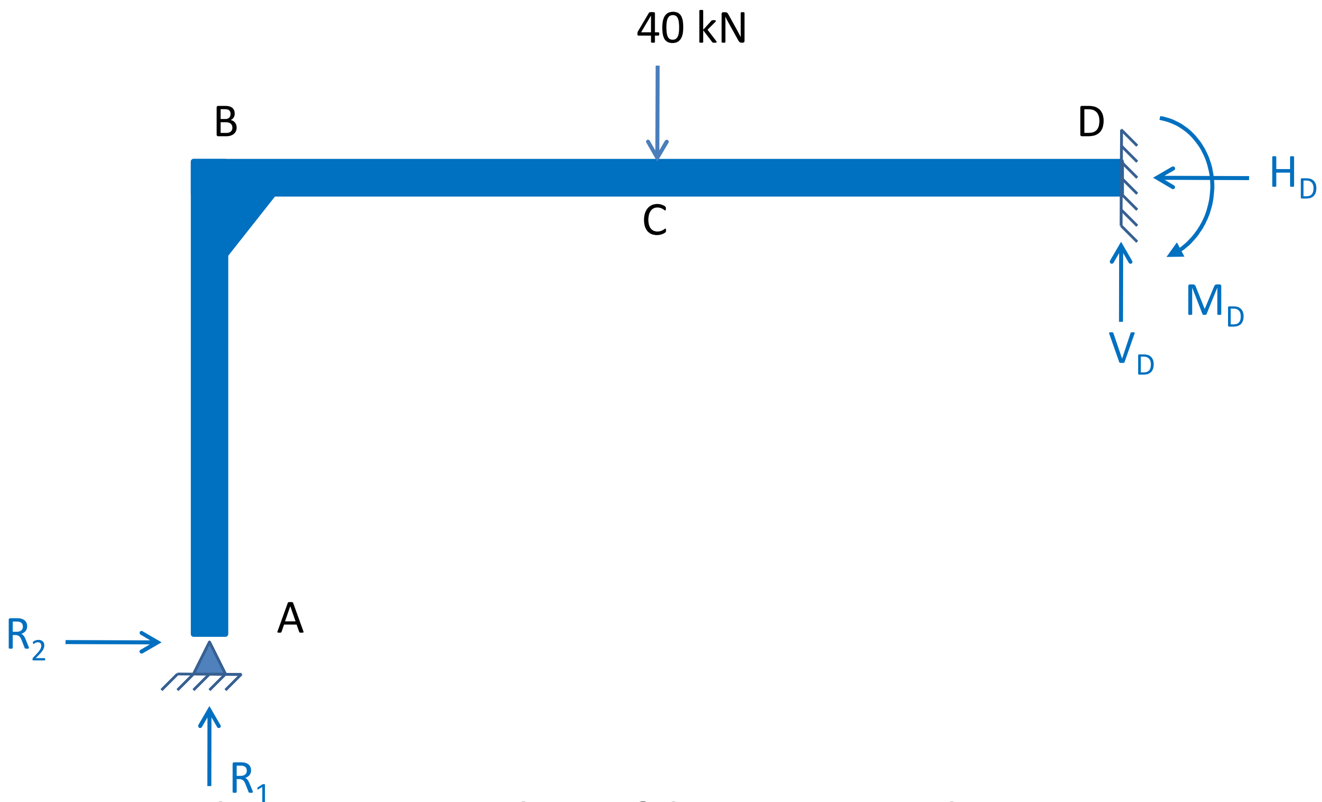


Solution

Let us choose R_1 and R_2, the reactions at A, to be the redundants.



Solution

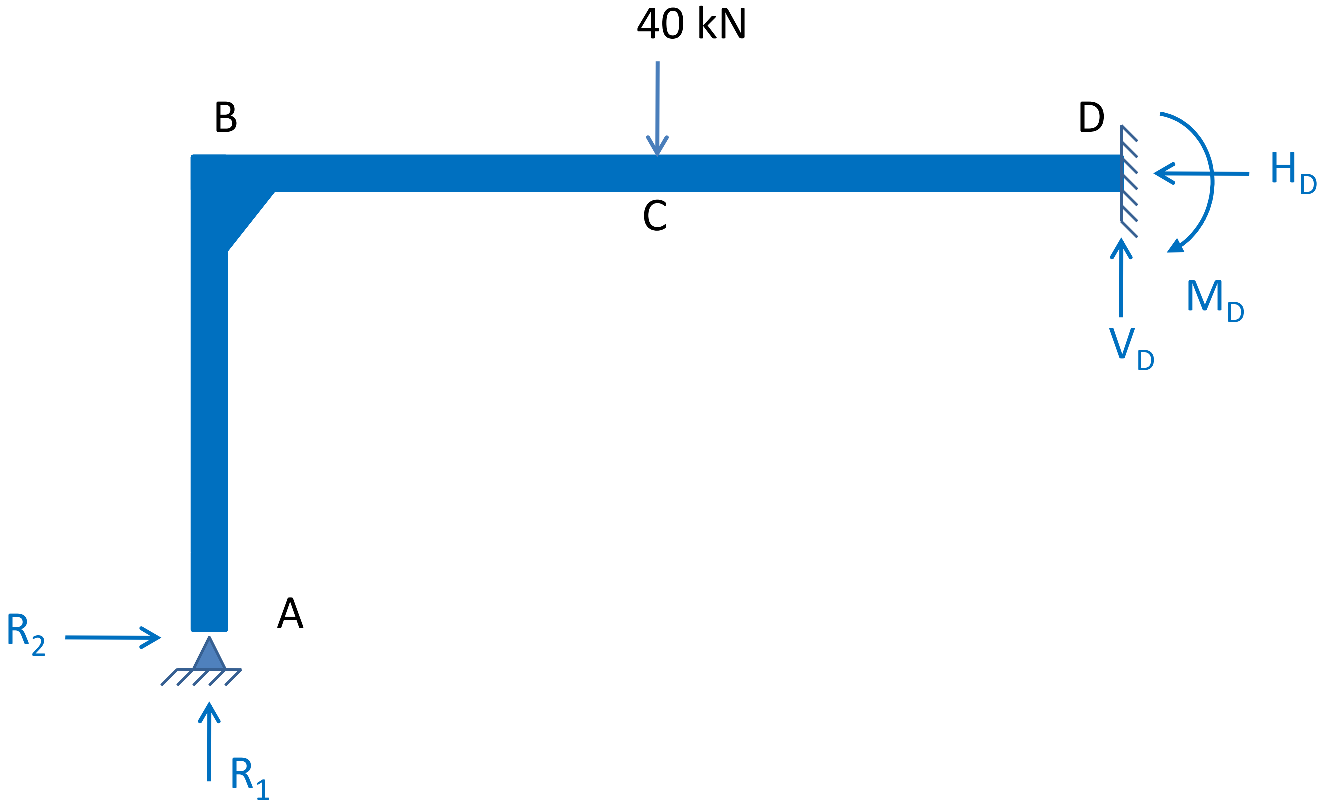


According to the principle of least work

$$\frac{\partial U}{\partial R_1} = \int_0^L \frac{\partial M}{\partial R_1}\frac{M}{EI}dx = 0 \qquad (1)$$

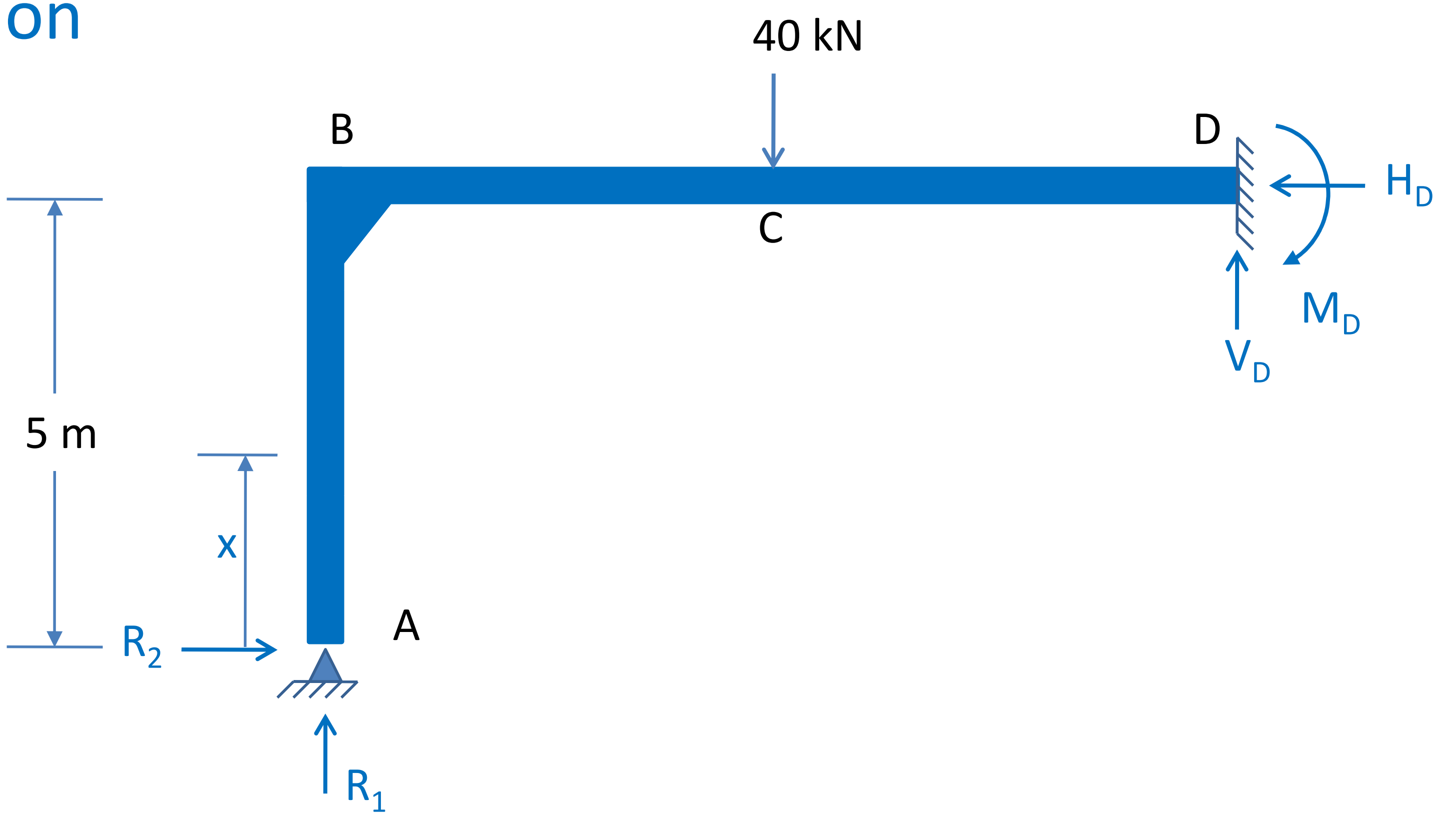
$$\frac{\partial U}{\partial R_2} = \int_0^L \frac{\partial M}{\partial R_2}\frac{M}{EI}dx = 0 \qquad (2)$$

Solution



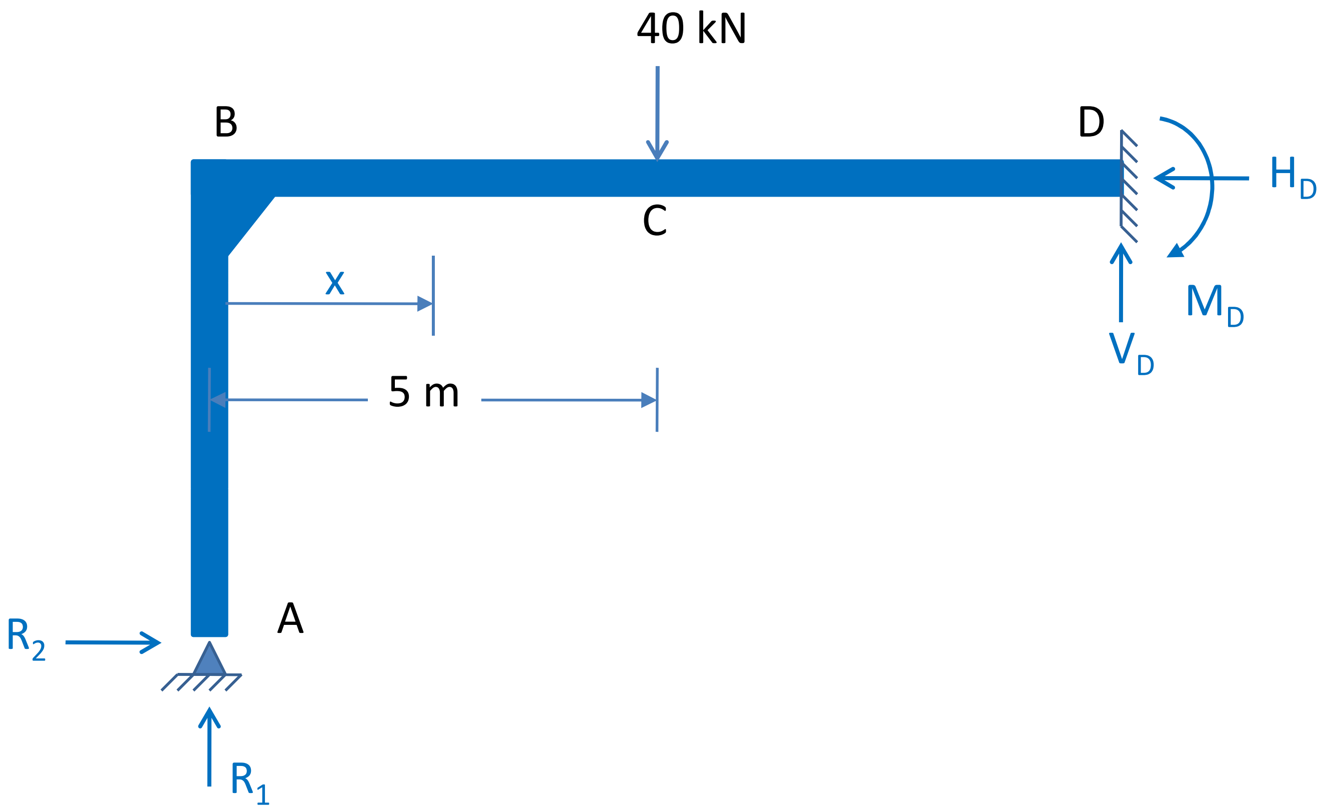
The expressions for moment and its derivative needed to solve Eq. (1) & (2) are listed in the table on next slide.

Solution



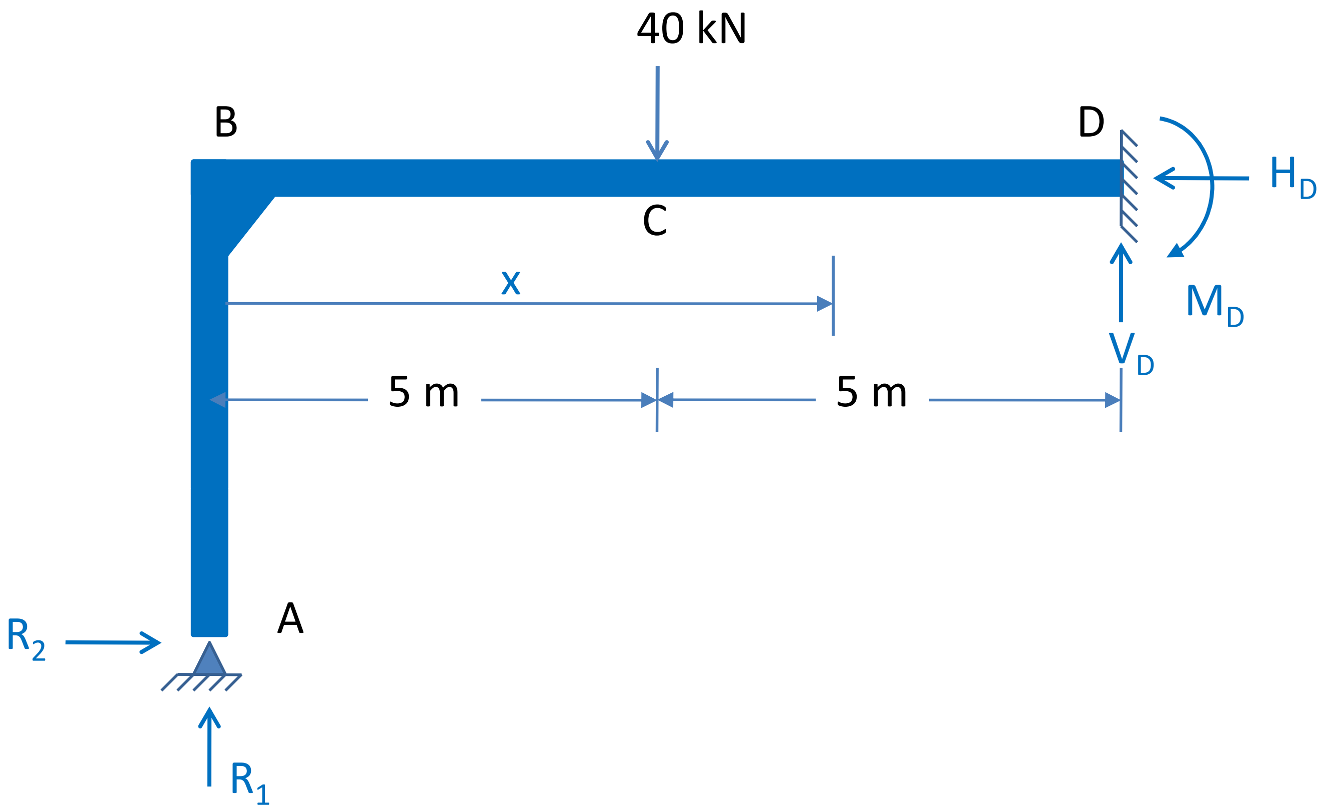
Segment	Origin	Limits	*M*	***∂M/∂R$_1$***	***∂M/∂R$_2$***
AB	A	0 – 5	$-R_2x$	0	-x

Solution



Segment	Origin	Limits	*M*	$\partial M/\partial R_1$	$\partial M/\partial R_2$
AB	A	0 – 5	$- R_2x$	0	-x
BC	B	0 – 5	$R_1x - 5R_2$	x	-5

Solution



Segment	Origin	Limits	*M*	$\partial M/\partial R_1$	$\partial M/\partial R_2$
AB	A	0 – 5	$-R_2x$	0	-x
BC	B	0 – 5	$R_1x - 5R_2$	x	-5
CD	B	5 – 10	$R_1x - 5R_2 - 40x + 200$	x	-5

Solution

Substitute these values into Eq. (1) & (2).

$$\frac{\partial U}{\partial R_1} = \int_0^L \frac{\partial M}{\partial R_1}\frac{M}{EI}dx = 0 \quad (1), \qquad \frac{\partial U}{\partial R_2} = \int_0^L \frac{\partial M}{\partial R_2}\frac{M}{EI}dx = 0 \quad (2)$$

Segment	Origin	Limits	*M*	∂M/∂R1	∂M/∂R2
AB	A	0 – 5	$-R_2x$	0	-x
BC	B	0 – 5	$R_1x - 5R_2$	x	-5
CD	B	5 – 10	$R_1x - 5R_2 - 40x + 200$	x	-5

$$\int_0^5 \left(R_1x^2 - 5R_2x\right)dx + \int_5^{10}\left(R_1x^2 - 5R_2x - 40x^2 + 200x\right)dx = 0$$

$$\int_0^5 R_2x^2dx + \int_0^5\left(-5R_1x + 25R_2\right)dx + \int_5^{10}\left(-5R_1x + 25R_2 + 200x - 1000\right)dx = 0$$

Solution

From which

$$333R_1 - 250R_2 - 4167 = 0$$

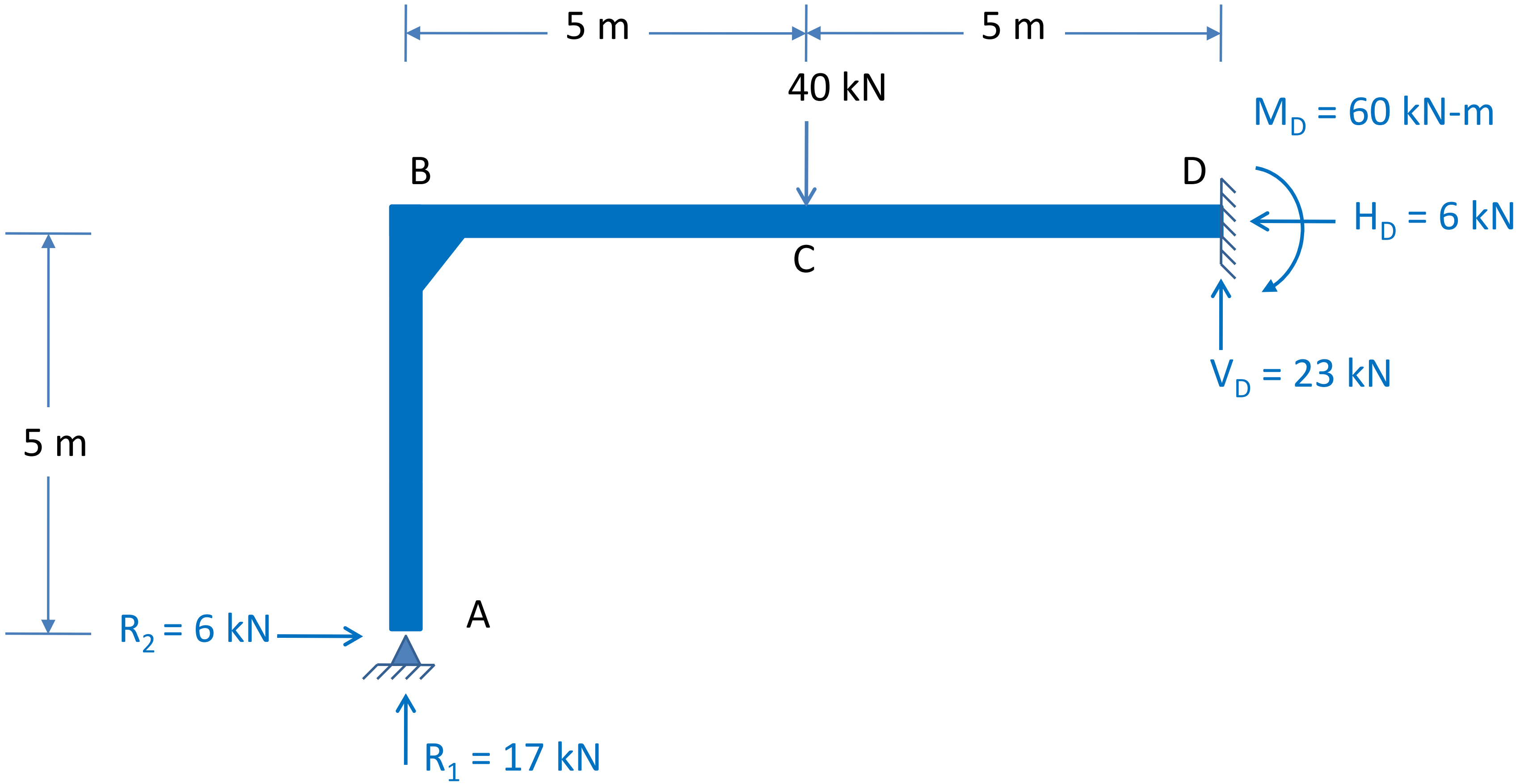
$$-250R_1 + 292R_2 + 2500 = 0$$

and

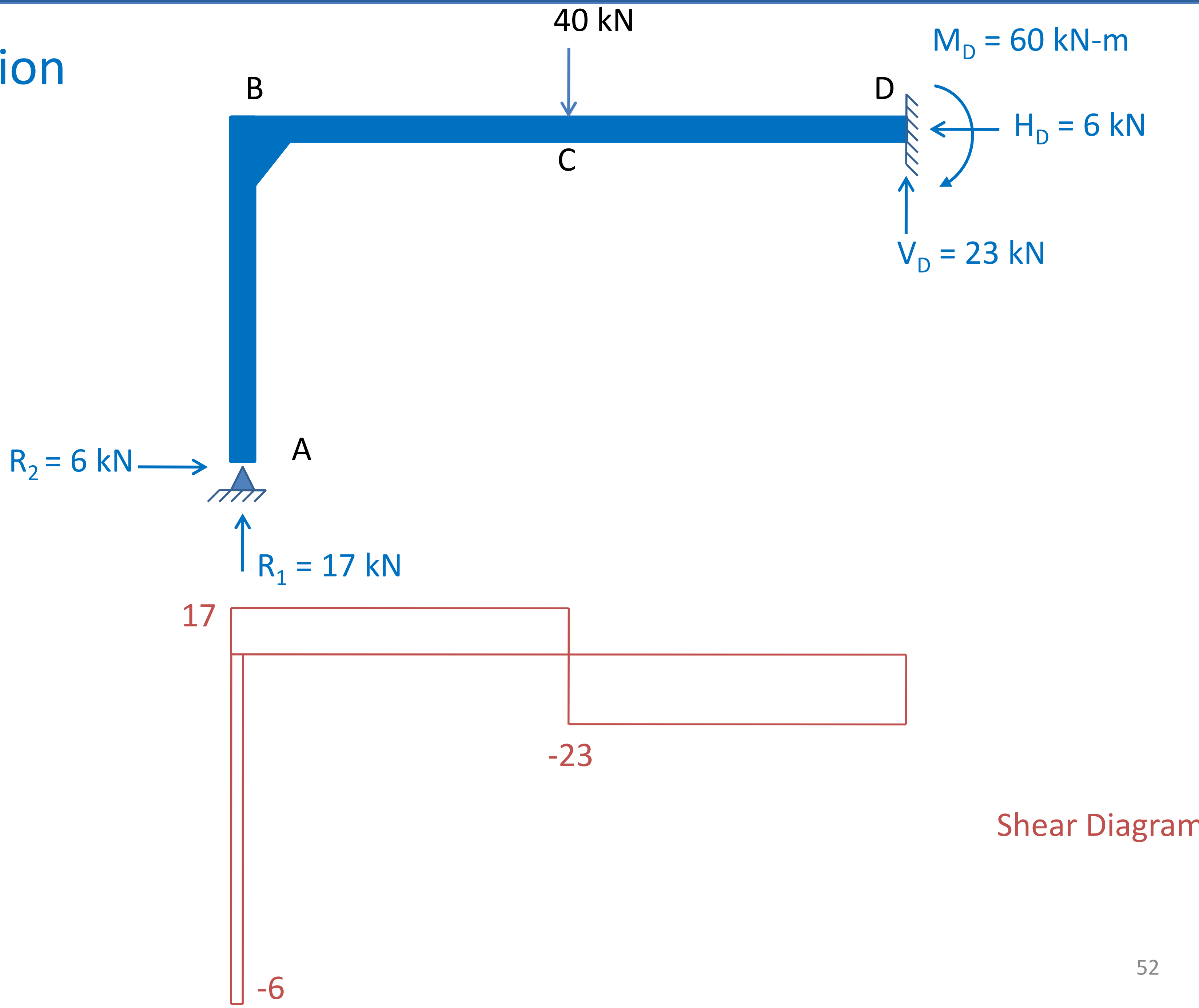
$$R_1 = 17.0\,kN \qquad \text{ANS}$$

$$R_2 = 6.0\,kN \qquad \text{ANS}$$

Solution

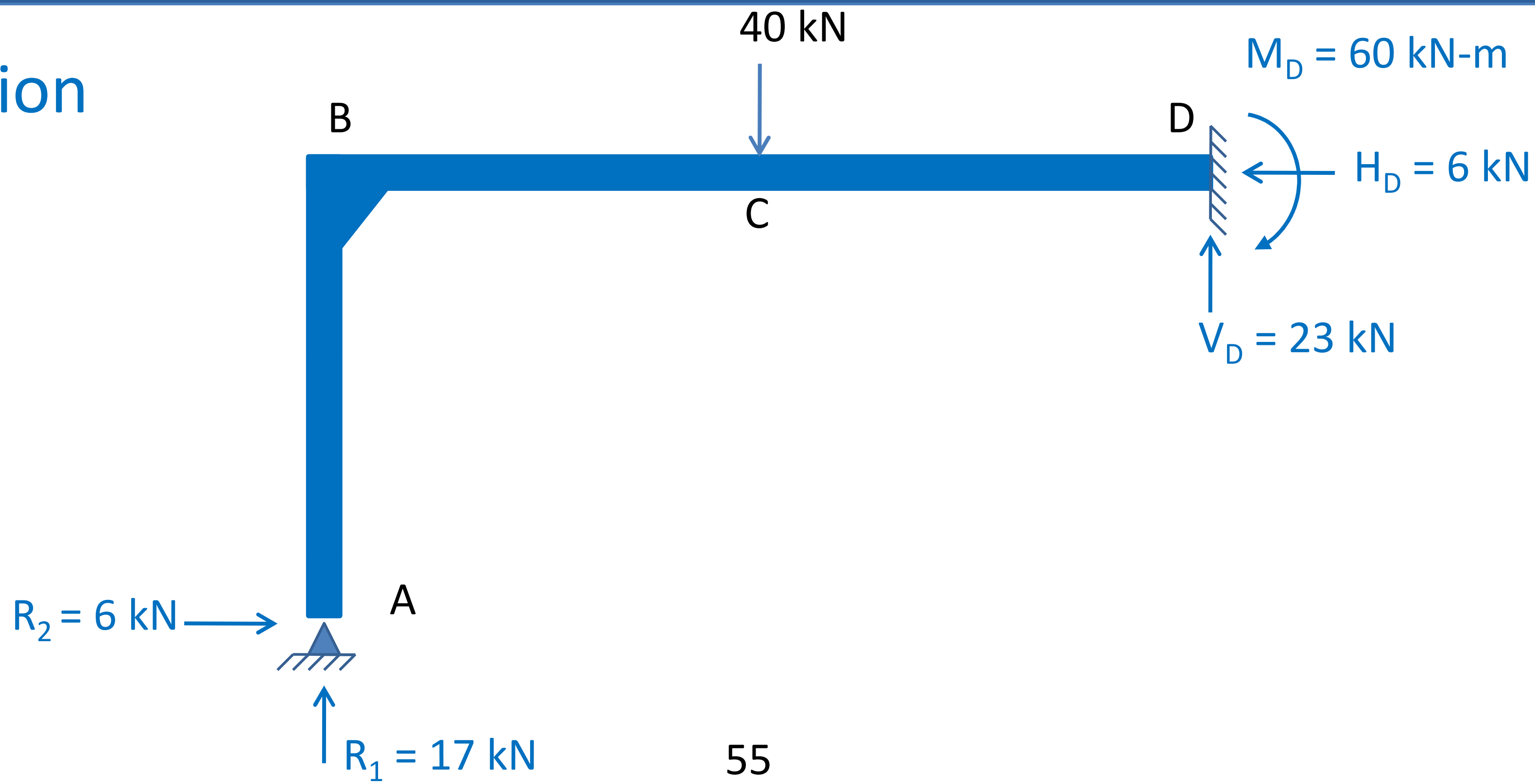


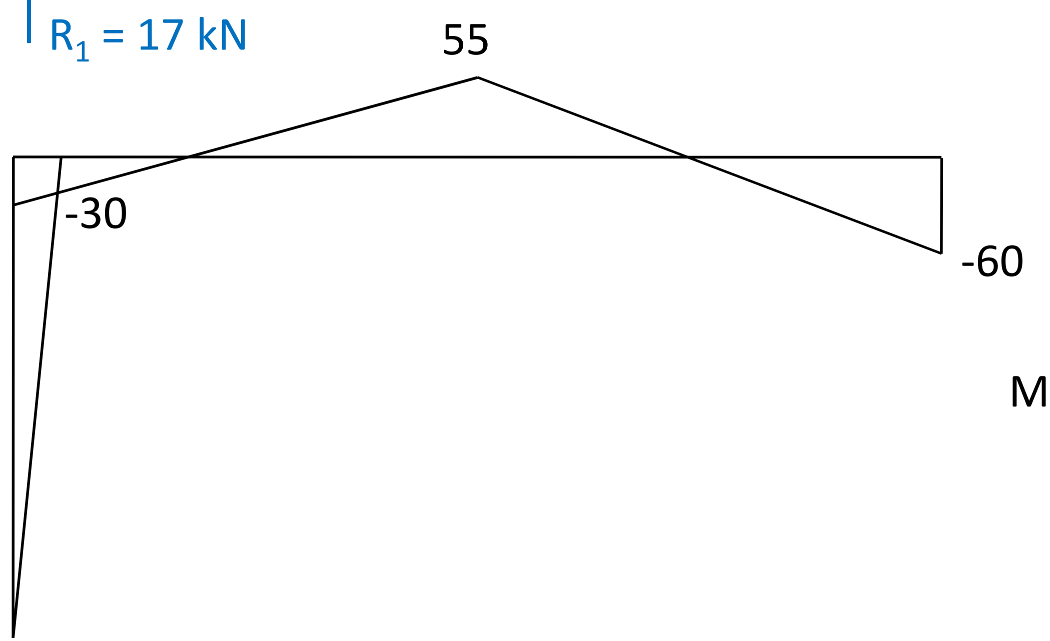
Solution



40 kN
M_D = 60 kN-m
B
D
H_D = 6 kN
C
V_D = 23 kN
A
R_2 = 6 kN
R_1 = 17 kN
17
-23
-6
Shear Diagram

Solution

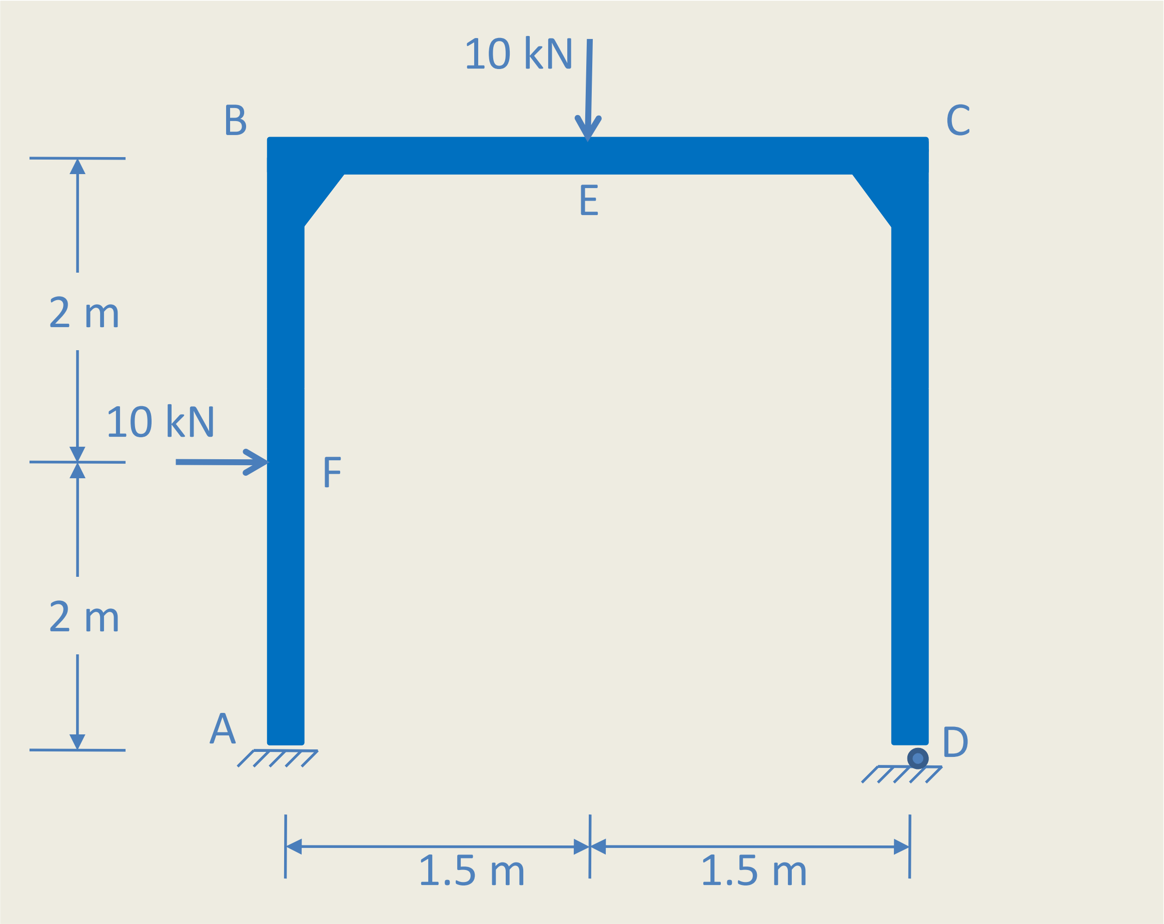




Moment Diagram

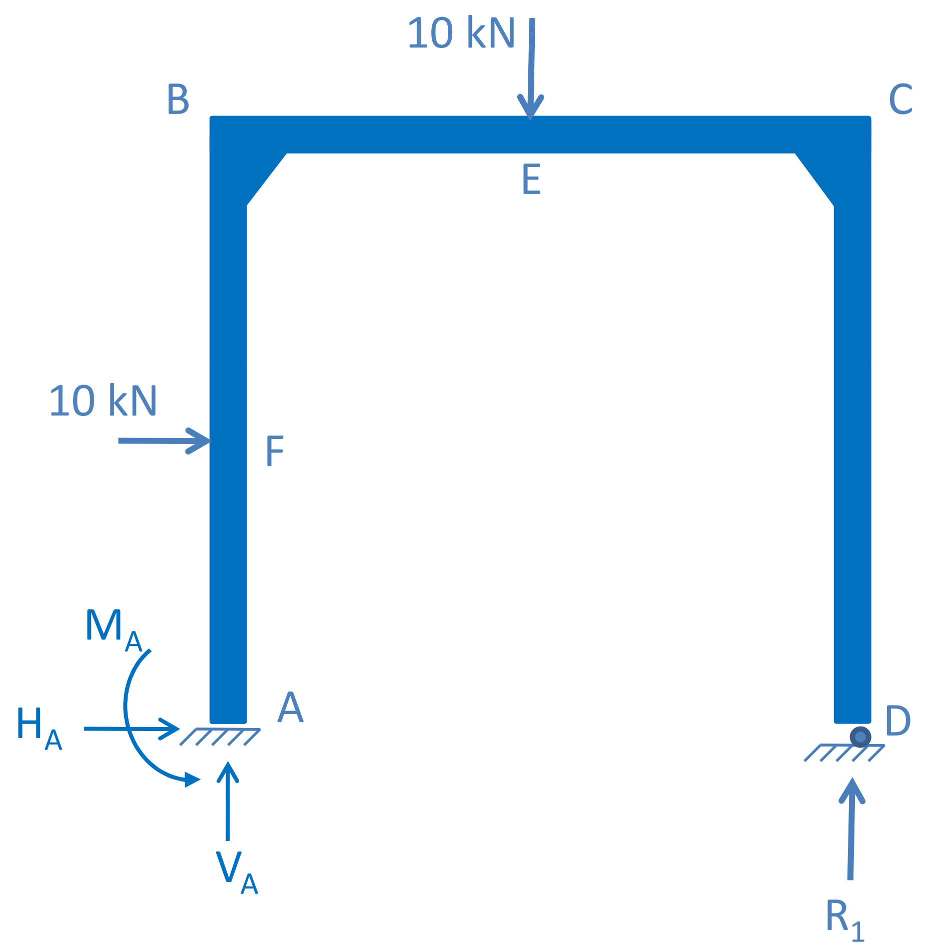
Example 5

Determine the reactions for the frame shown in Fig., by the method of least work. *EI* is constant.



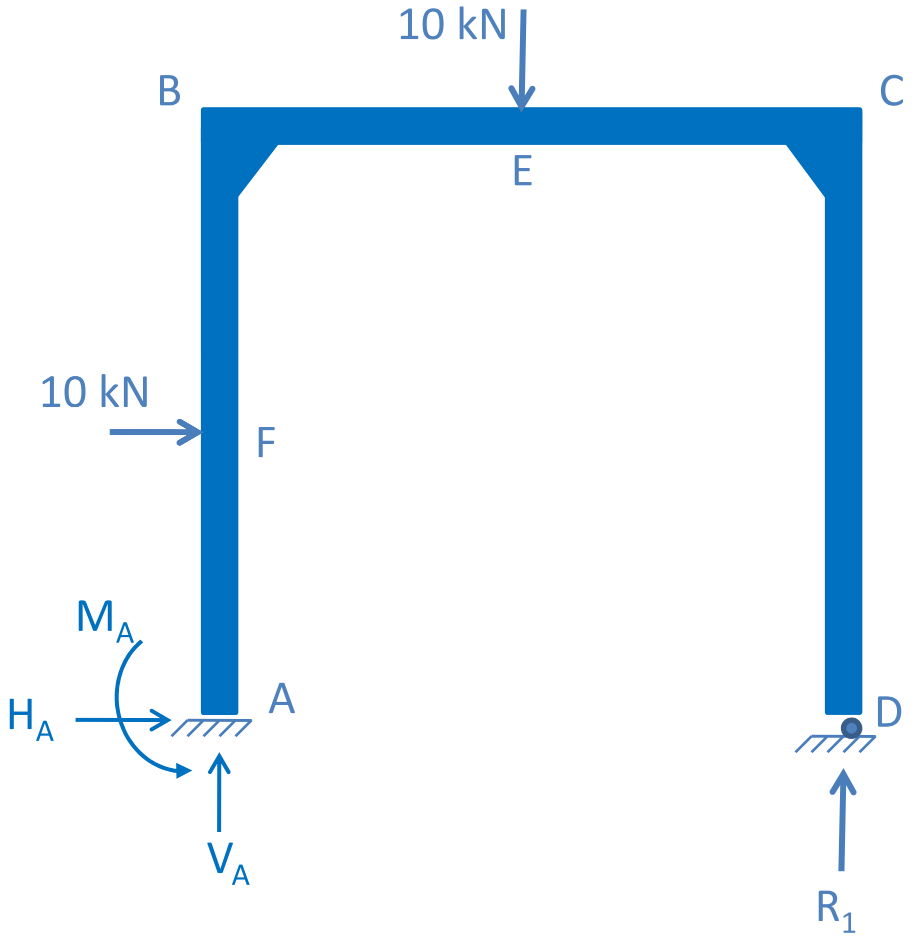
Solution

The structure is indeterminate to the first degree. It has single redundant reaction.

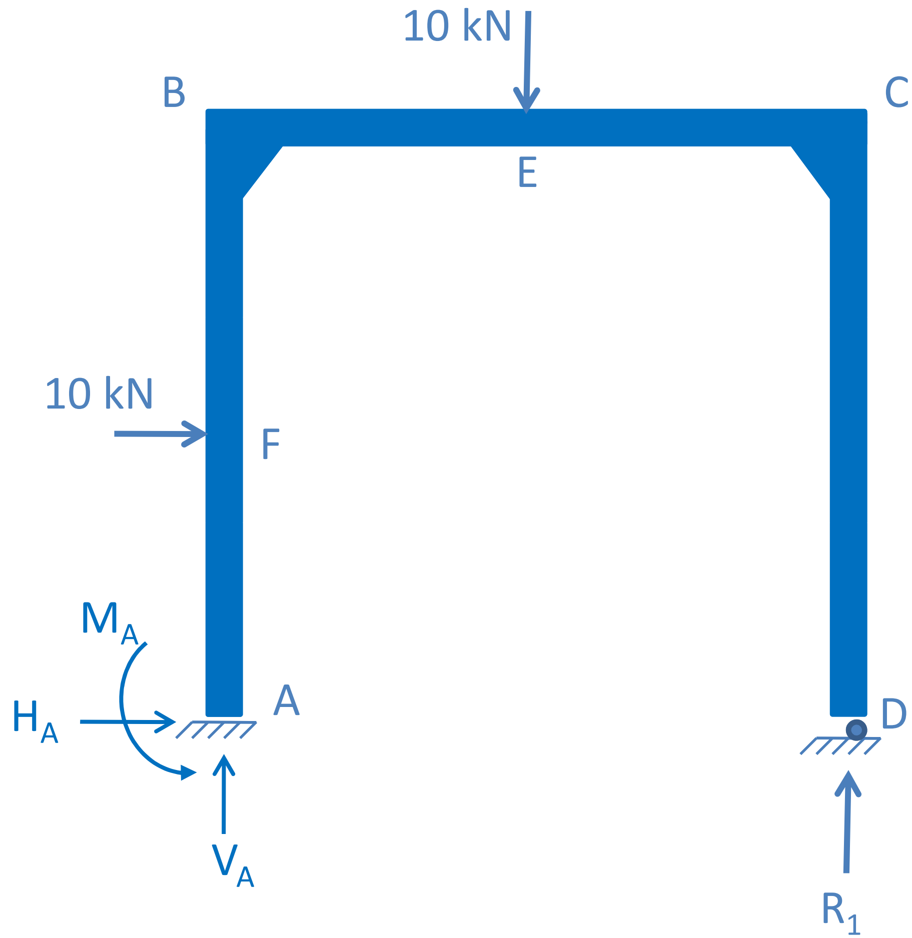


Solution

Let us choose R_1, the reaction at D, to be the redundant.



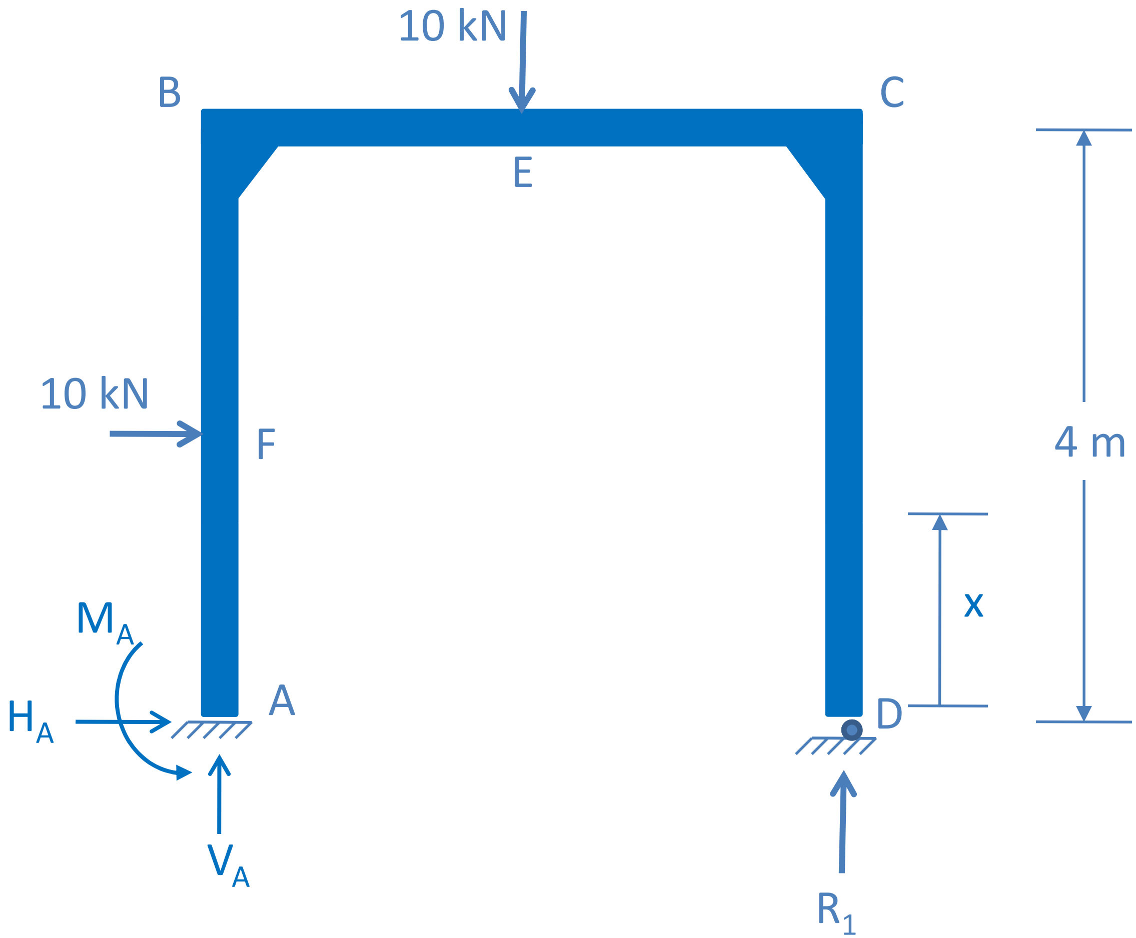
Solution



According to the principle of least work

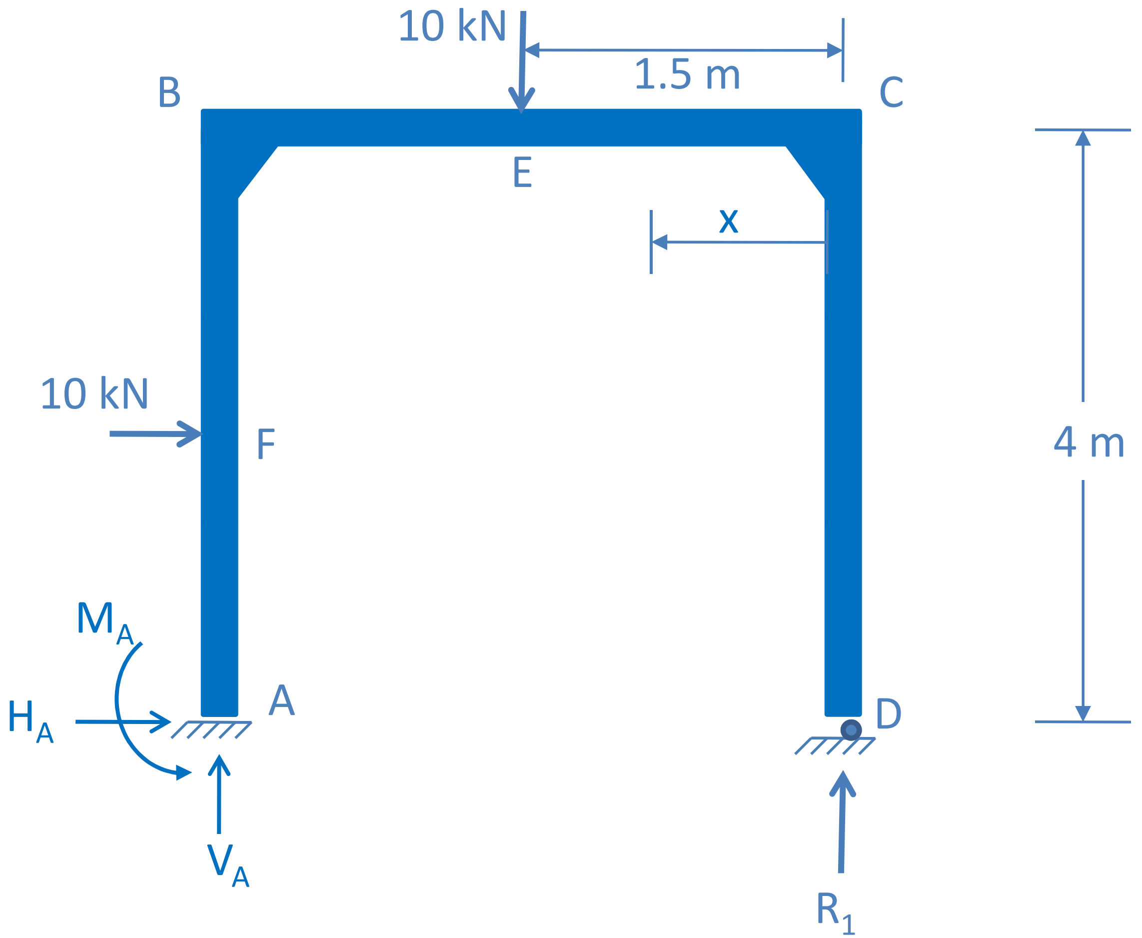
$$\frac{\partial U}{\partial R_1} = \int_0^L \frac{\partial M}{\partial R_1} \frac{M}{EI} dx = 0 \qquad (1)$$

Solution



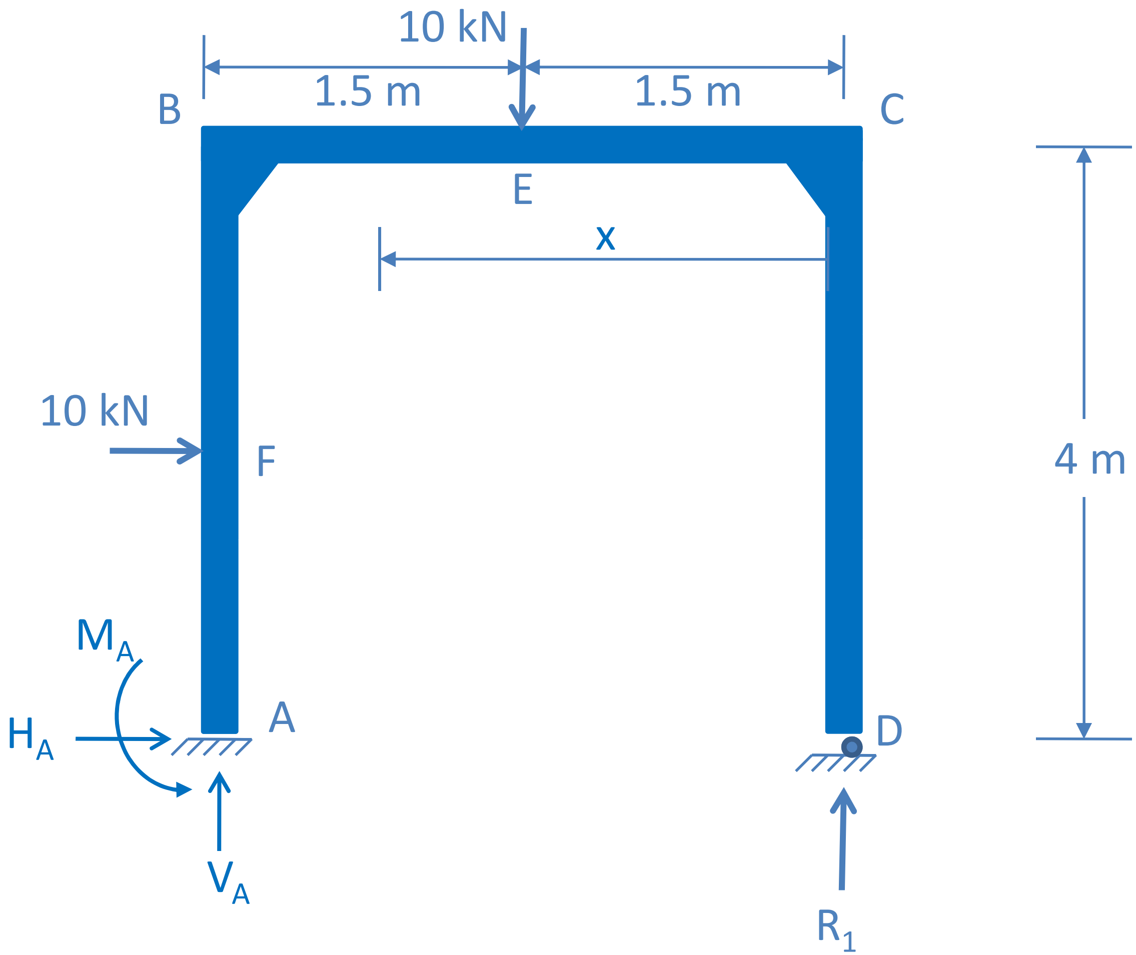
Segment	Origin	Limits	*M*	*∂M/∂R₁*
DC	D	0 – 4	0	0

Solution



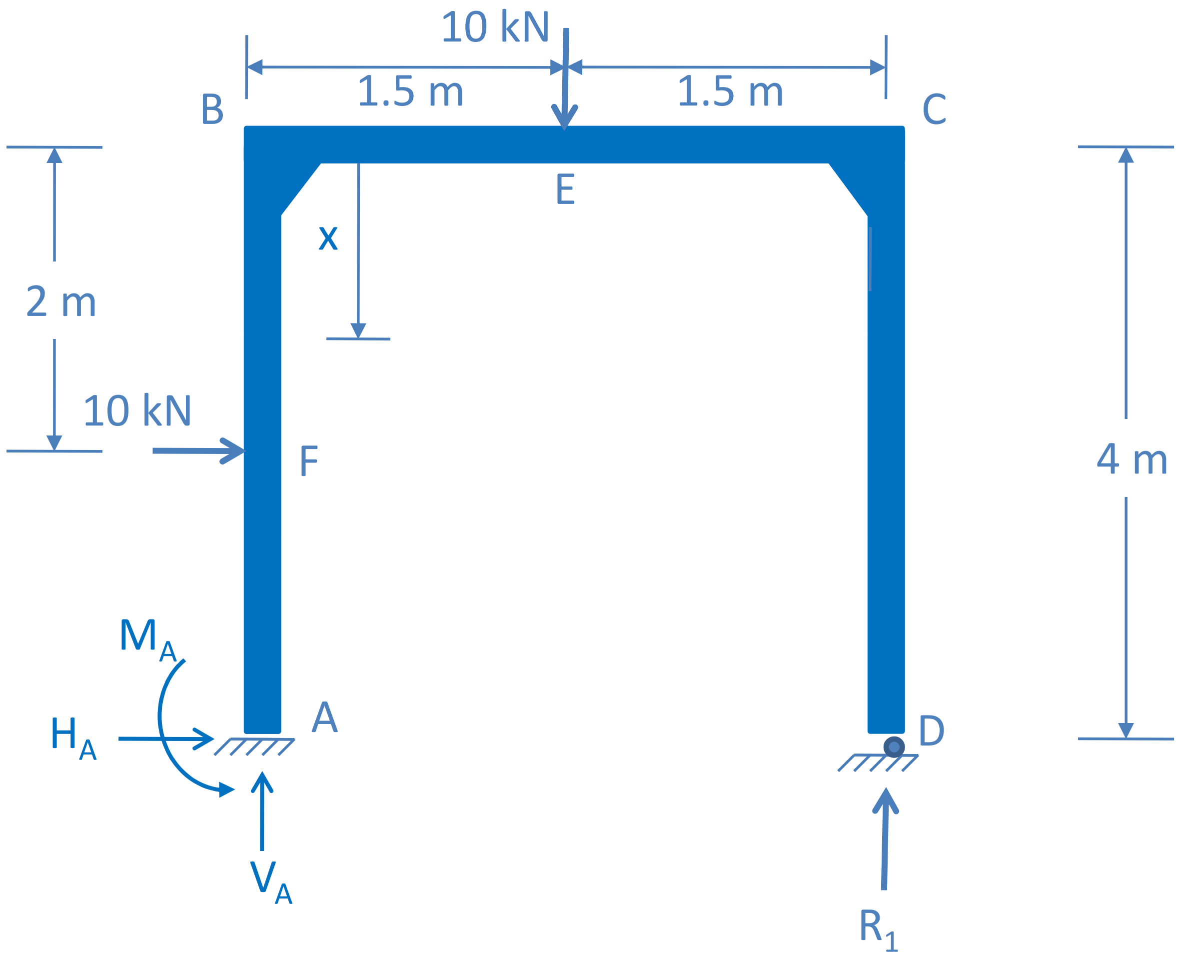
Segment	Origin	Limits	*M*	$\partial M/\partial R_1$
DC	D	0 – 4	0	0
CE	C	0 – 1.5	$R_1.x$	x

Solution



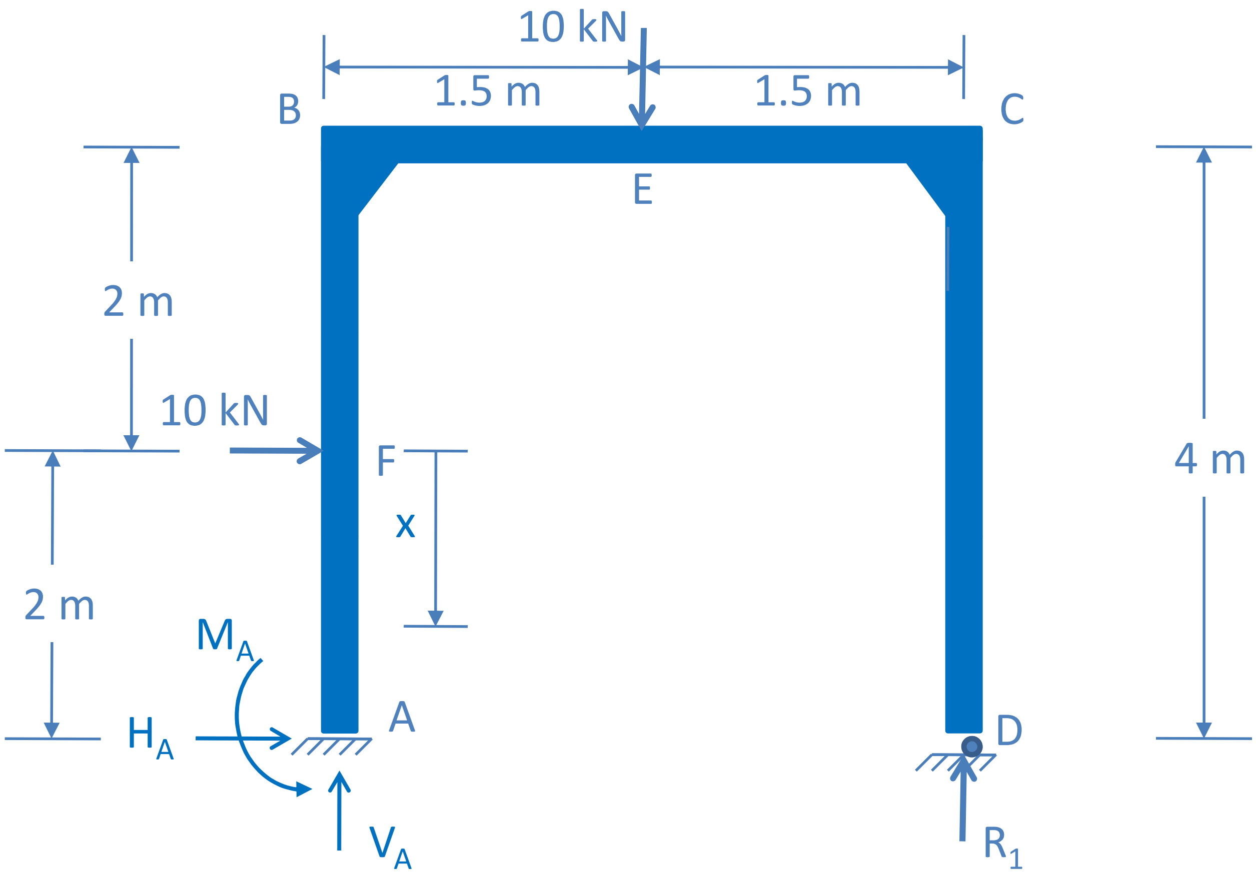
Segment	Origin	Limits	*M*	$\partial M/\partial R_1$
DC	D	0 – 4	0	0
CE	C	0 – 1.5	$R_1.x$	x
EB	C	1.5 – 3	$R_1.x - 10(x - 1.5)$	x

Solution



Segment	Origin	Limits	***M***	$\partial M/\partial R_1$
DC	D	0 – 4	0	0
CE	C	0 – 1.5	$R_1.x$	x
EB	C	1.5 – 3	$R_1.x - 10(x - 1.5)$	x
BF	B	0 – 2	$3R_1 - 10(1.5)$	3

Solution



Segment	Origin	Limits	*M*	*∂M/∂R₁*
DC	D	0 – 4	0	0
CE	C	0 – 1.5	$R_1.x$	x
EB	C	1.5 – 3	$R_1.x - 10(x - 1.5)$	x
BF	B	0 – 2	$3R_1 - 10(1.5)$	3
FA	F	0 – 2	$3R_1 - 10(1.5) - 10x$	3

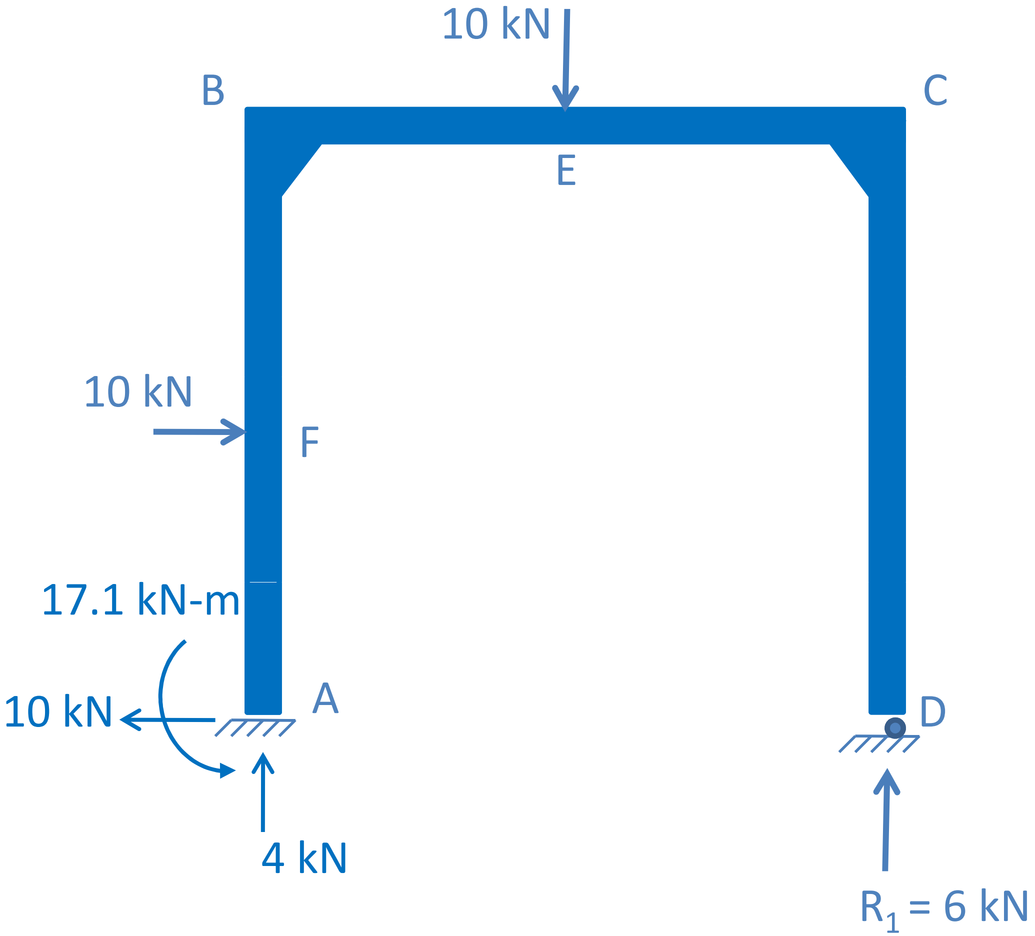
Solution

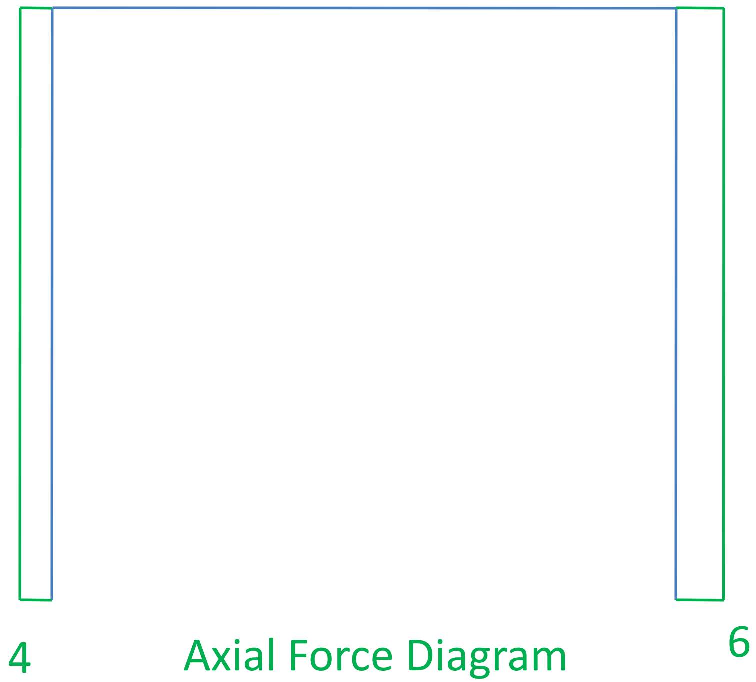
Segment	Origin	Limits	*M*	*∂M/∂R₁*
DC	D	0 – 4	0	0
CE	C	0 – 1.5	$R_1.x$	x
EB	C	1.5 – 3	$R_1.x - 10(x - 1.5)$	x
BF	B	0 – 2	$3R_1 - 10(1.5)$	3
FA	F	0 – 2	$3R_1 - 10(1.5) - 10x$	3

$$\frac{1}{EI}\int_0^{1.5} R_1 x^2 dx + \frac{1}{EI}\int_{1.5}^{3.0} [(R_1 - 10)x + 15]x dx + \frac{1}{EI}\int_0^2 (9R_1 - 45)dx + \frac{1}{EI}\int_0^2 (9R_1 - 45 - 30x)dx = 0$$

$$R_1 = 5.958\, kN \cong 6\, kN$$

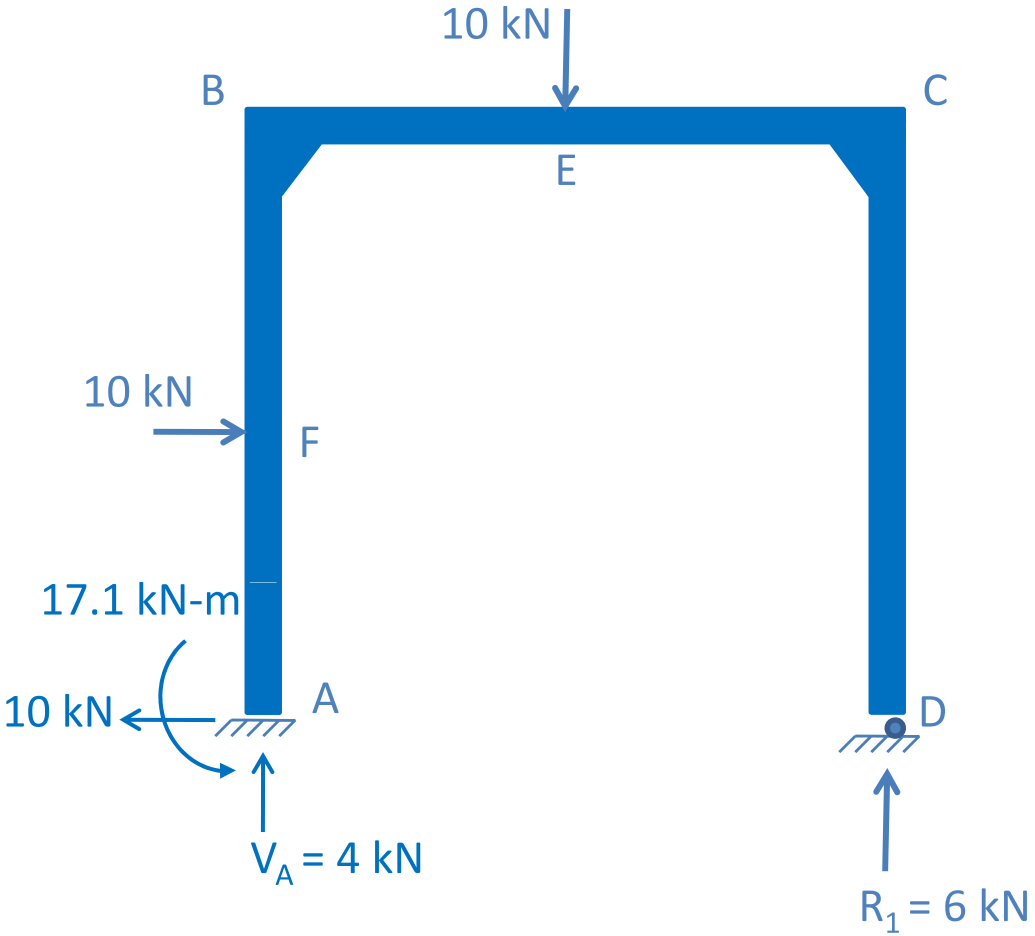
Solution

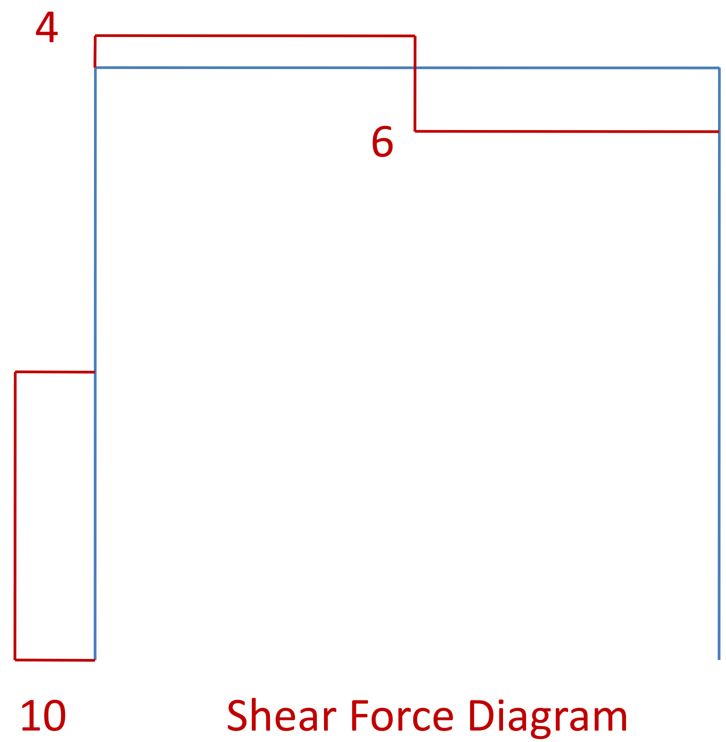




Axial Force Diagram

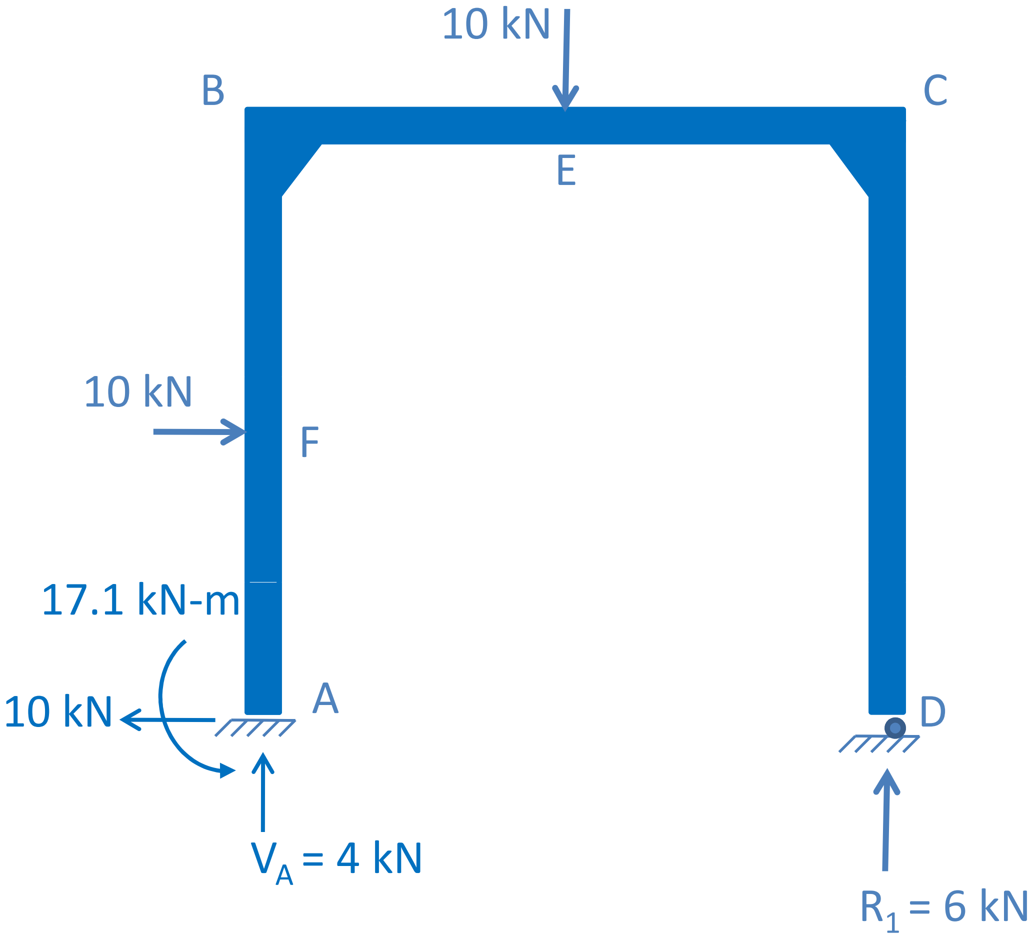
Solution

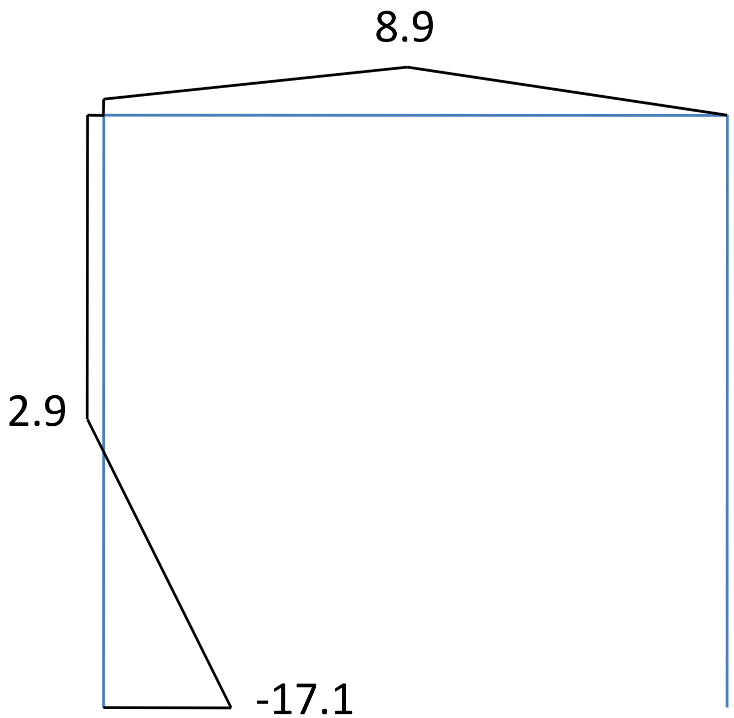




Shear Force Diagram

Solution

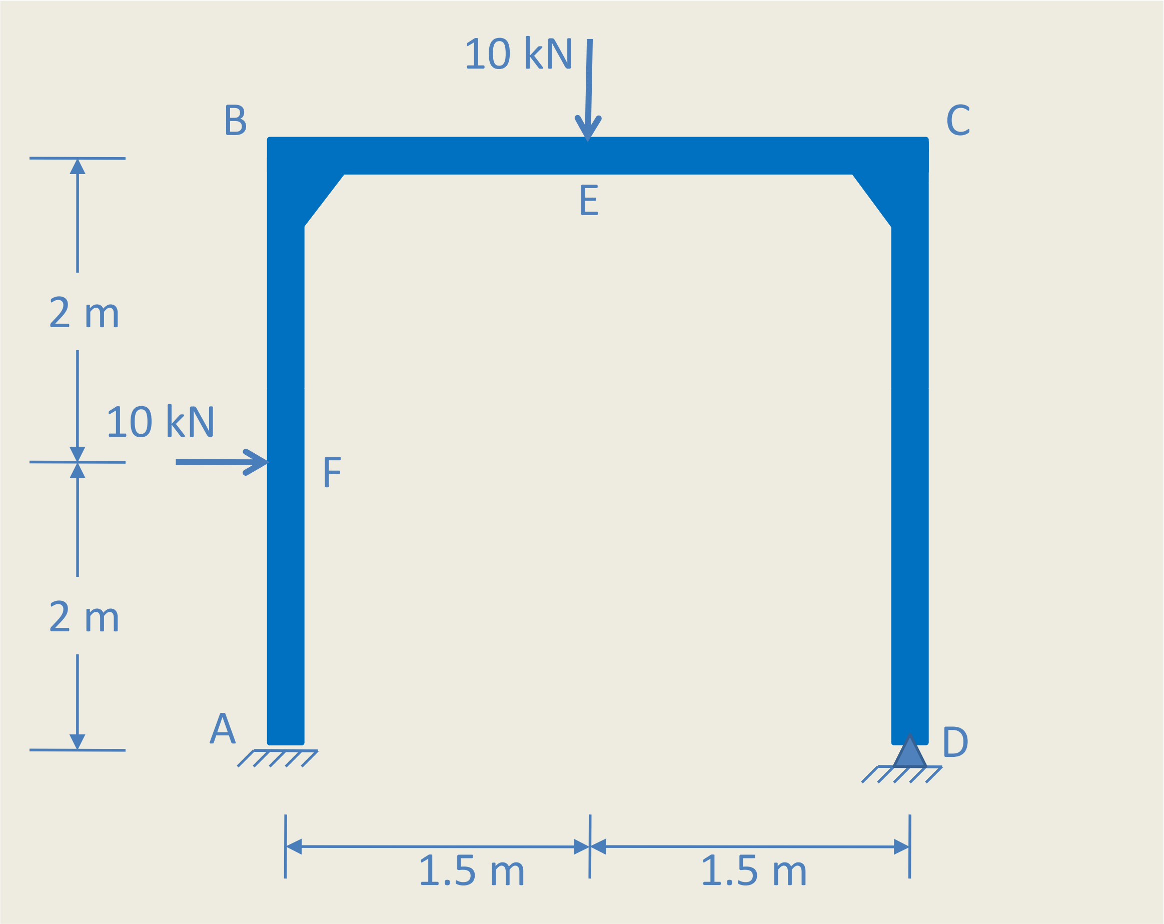




Bending Moment Diagram

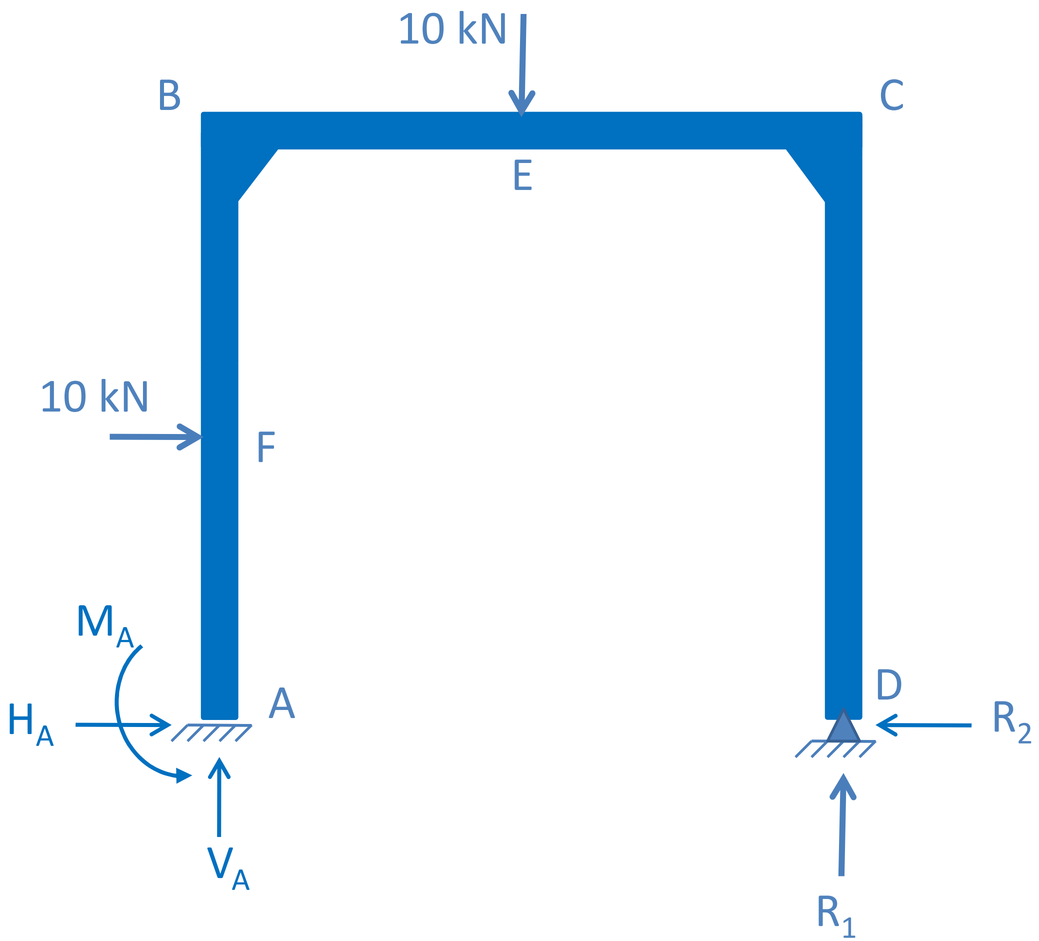
Example 6

Determine the reactions for the frame shown in Fig., by the method of least work. *EI* is constant.



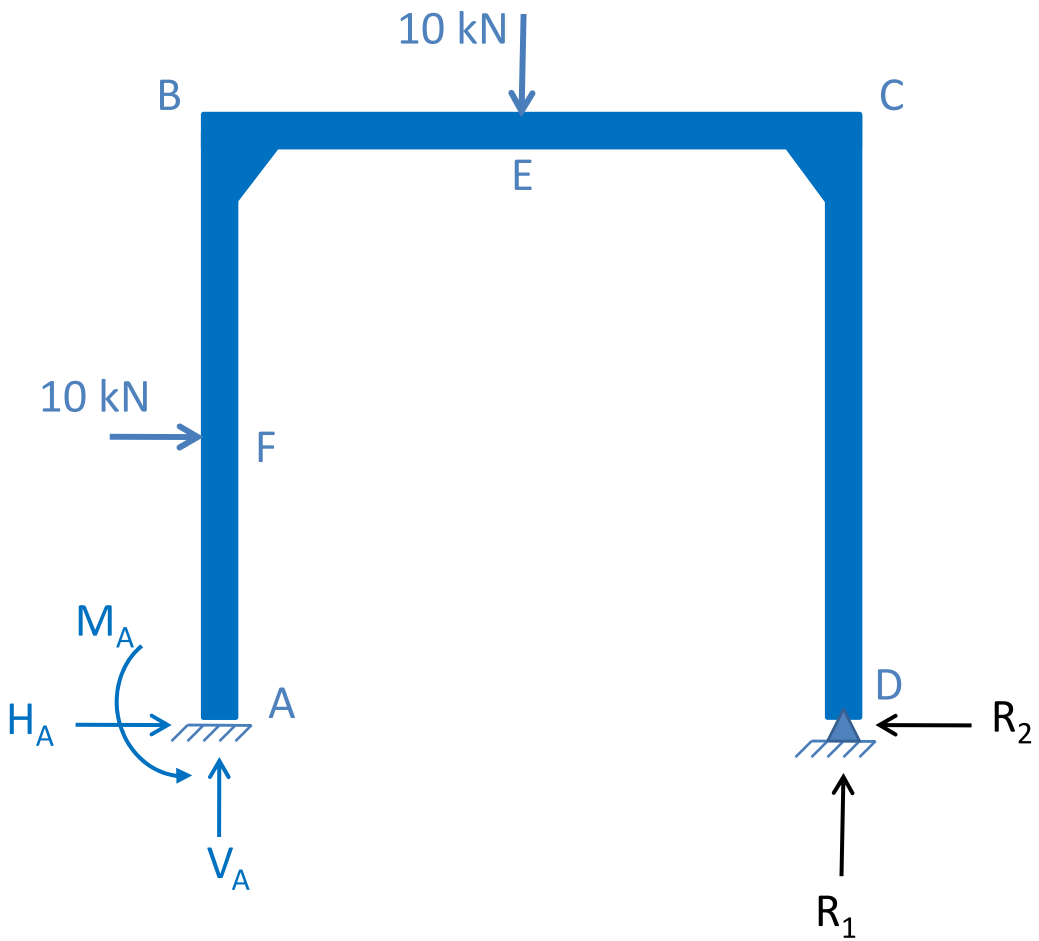
Solution

The structure is indeterminate to the second degree. It has two redundant reactions.



Solution

Let us choose R_1, R_2, the reaction at D, to be the redundant.

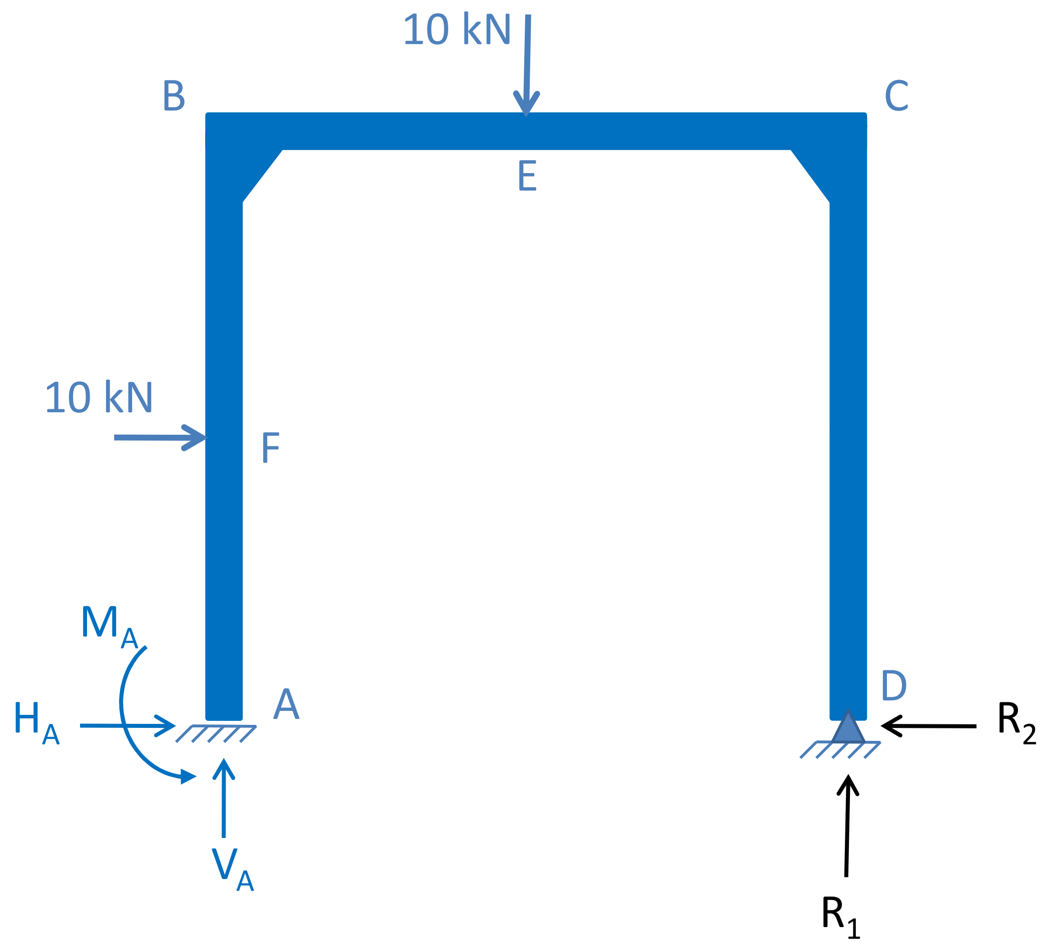


Solution

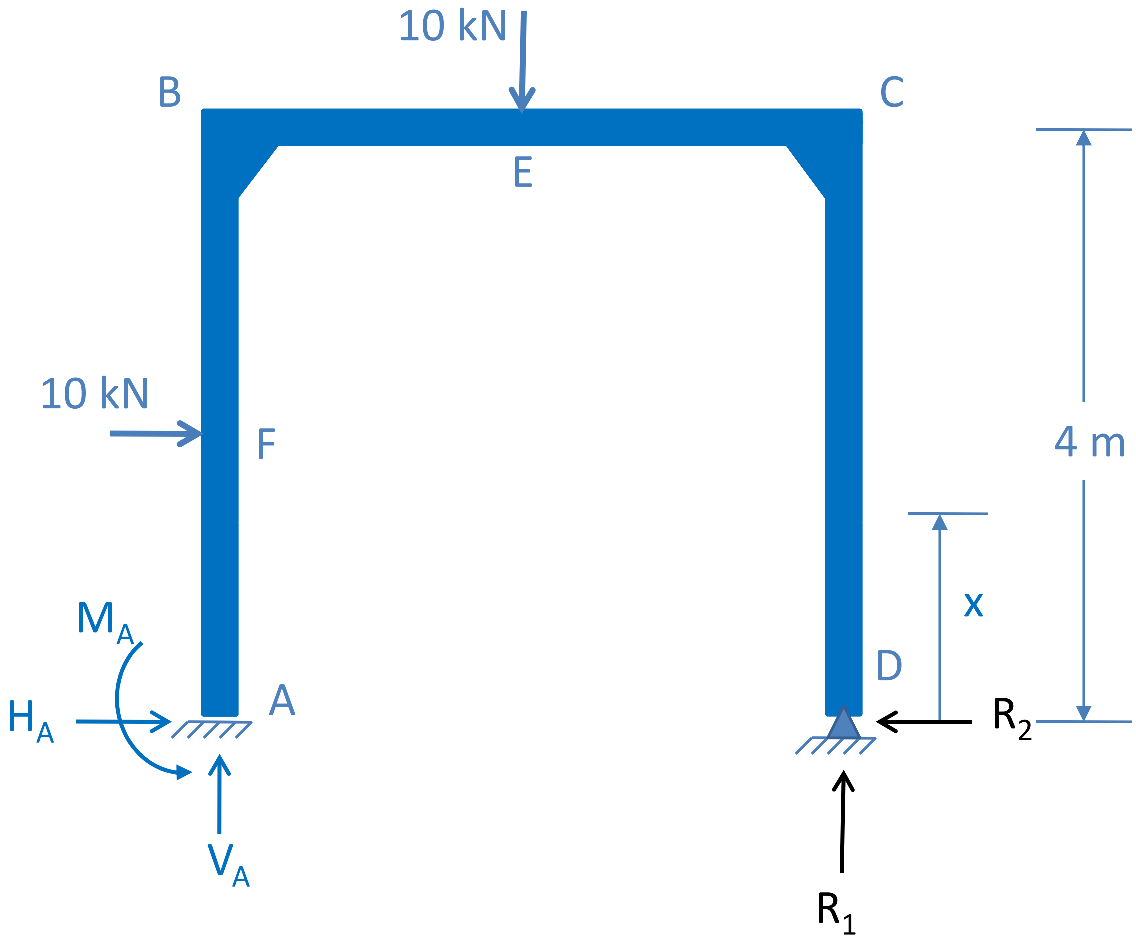
According to the Principle of Least Work

$$\frac{\partial U}{\partial R_1} = \int_0^L \frac{\partial M}{\partial R_1}\frac{M}{EI}dx = 0$$

$$\frac{\partial U}{\partial R_2} = \int_0^L \frac{\partial M}{\partial R_2}\frac{M}{EI}dx = 0$$

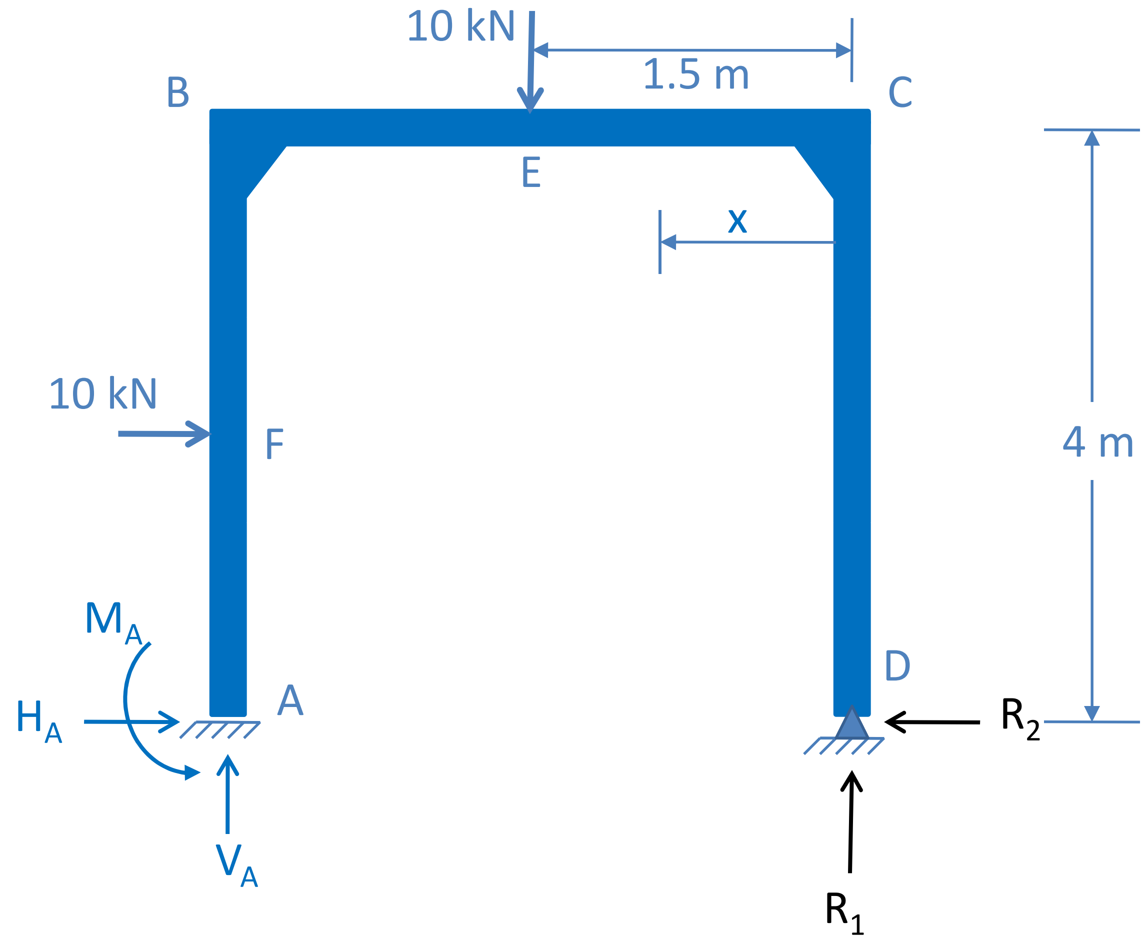


Solution



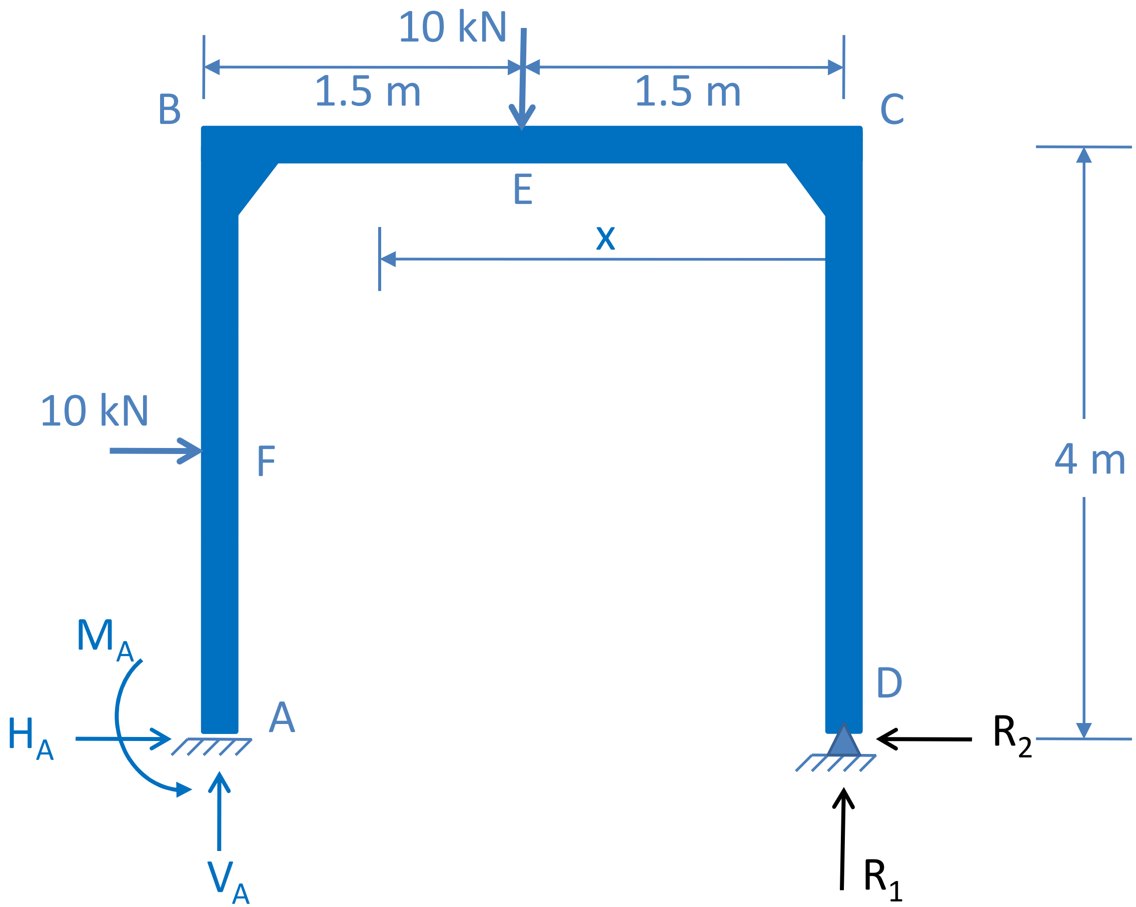
Segment	Origin	Limits	*M*	*∂M/∂R₁*	*∂M/∂R₂*
DC	D	0 – 4	$-R_2.x$	0	-x

Solution



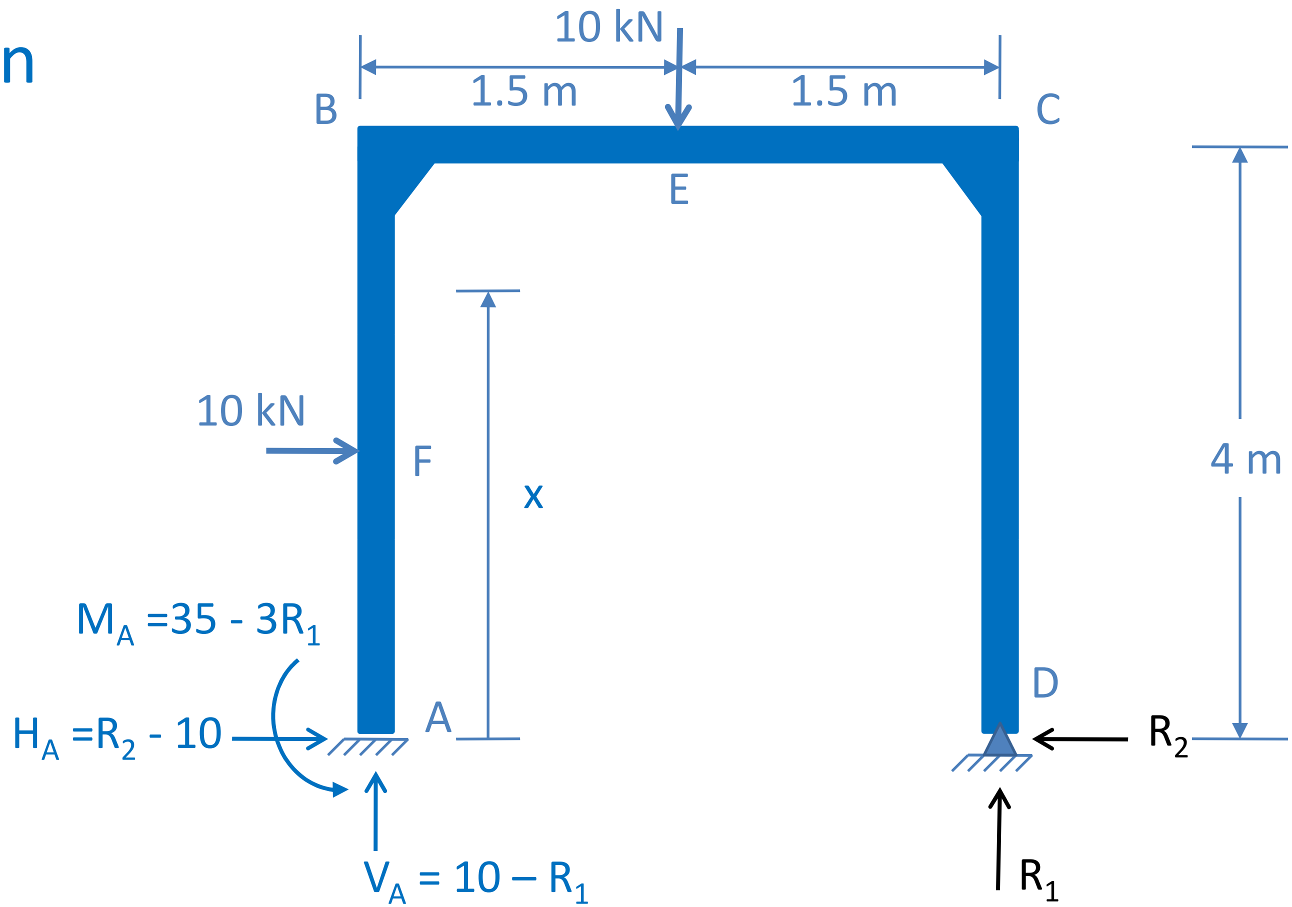
Segment	Origin	Limits	*M*	*∂M/∂R₁*	*∂M/∂R₂*
DC	D	0 – 4	$-R_2.x$	0	-x
CE	C	0 – 1.5	$-4R_2 + R_1.x$	x	-4

Solution



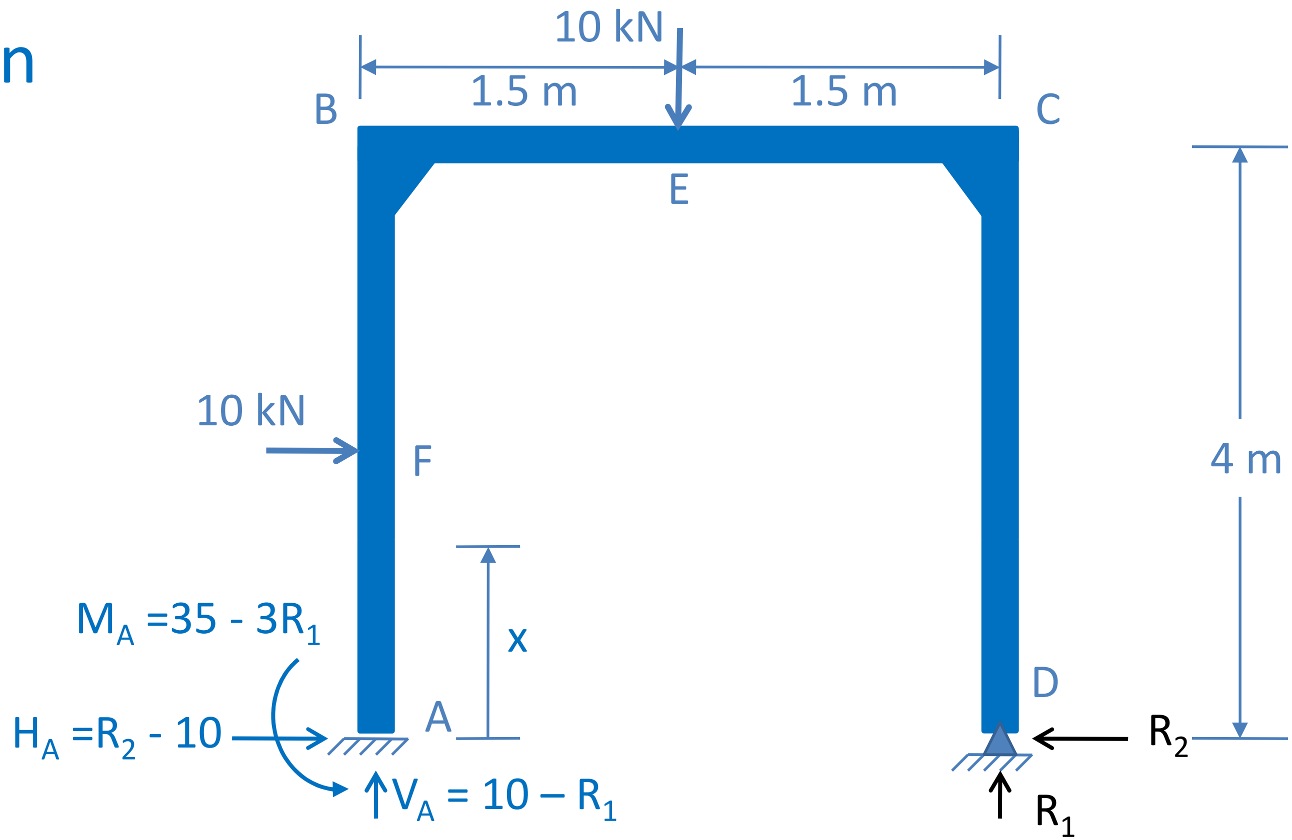
Segment	Origin	Limits	*M*	*∂M/∂R₁*	*∂M/∂R₂*
DC	D	0 – 4	$-R_2.x$	0	-x
CE	C	0 – 1.5	$-4R_2 + R_1.x$	x	-4
EB	C	1.5 – 3.0	$-4R_2 + R_1.x -10(x-1.5)$	x	-4

Solution



Segment	Origin	Limits	M	$\partial M/\partial R_1$	$\partial M/\partial R_2$
DC	D	0 – 4	$-R_2.x$	0	-x
CE	C	0 – 1.5	$-4R_2 + R_1.x$	x	-4
EB	C	1.5 – 3.0	$-4R_2 + R_1.x -10(x-1.5)$	x	-4
FB	A	2 – 4	$-35 + 3R_1 – (R_2 – 10)x – 10(x – 2)$	3	-x

Solution



Segment	Origin	Limits	*M*	$\partial M/\partial R_1$	$\partial M/\partial R_2$
DC	D	0 – 4	$-R_2.x$	0	-x
CE	C	0 – 1.5	$-4R_2 + R_1.x$	x	-4
EB	C	1.5 – 3.0	$-4R_2 + R_1.x - 10(x - 1.5)$	x	-4
FB	A	2 – 4	$-35 + 3R_1 - (R_2 - 10)x - 10(x - 2)$	3	-x
AF	A	0 – 2	$-35 + 3R_1 - (R_2 - 10)x$	3	-x

Solution

Segment	Origin	Limits	*M*	$\partial M/\partial R_1$	$\partial M/\partial R_2$
DC	D	0 – 4	$-R_2.x$	0	-x
CE	C	0 – 1.5	$-4R_2 + R_1.x$	x	-4
EB	C	1.5 – 3.0	$-4R_2 + R_1.x - 10(x - 1.5)$	x	-4
FB	A	2 – 4	$-35 + 3R_1 - (R_2 - 10)x - 10(x - 2)$	3	-x
AF	A	0 – 2	$-35 + 3R_1 - (R_2 - 10)x$	3	-x

$$\frac{\partial U}{\partial R_1} = \int_0^L \frac{\partial M}{\partial R_1}\frac{M}{EI}dx = 0$$

$$\frac{1}{EI}\int_0^{1.5}(-4R_2 + R_1x)xdx + \frac{1}{EI}\int_{1.5}^{3.0}[-4R_2 + R_1x - 10(x - 1.5)]xdx$$
$$+\frac{1}{EI}\int_2^4(-35 + 3R_1 - (R_2 - 10)x - 10(x-2))3dx + \frac{1}{EI}\int_0^2(-35 + 3R_1 - (R_2 - 10)x)3dx = 0$$

$$45R_1 - 42R_2 - 268.125 = 0$$

Solution

Segment	Origin	Limits	*M*	$\partial M/\partial R_1$	$\partial M/\partial R_2$
DC	D	0 – 4	$-R_2.x$	0	-x
CE	C	0 – 1.5	$-4R_2 + R_1.x$	x	-4
EB	C	1.5 – 3.0	$-4R_2 + R_1.x - 10(x - 1.5)$	x	-4
FB	A	2 – 4	$-35 + 3R_1 - (R_2 - 10)x - 10(x - 2)$	3	-x
AF	A	0 – 2	$-35 + 3R_1 - (R_2 - 10)x$	3	-x

$$\frac{\partial U}{\partial R_2} = \int_0^L \frac{\partial M}{\partial R_2}\frac{M}{EI}dx = 0$$

$$\frac{1}{EI}\int_0^4 (-R_2x)(-x)dx + \frac{1}{EI}\int_0^{1.5}(-4R_2 + R_1x)(-4)dx + \frac{1}{EI}\int_{1.5}^{3.0}[-4R_2 + R_1x - 10(x-1.5)](-4)dx$$

$$+\frac{1}{EI}\int_2^4 (-35 + 3R_1 - (R_2 - 10)x - 10(x-2))(-x)dx + \frac{1}{EI}\int_0^2 (-35 + 3R_1 - (R_2 - 10)x)(-x)dx = 0$$

$$-42R_1 + 90.67R_2 + 178.33 = 0$$

Solution

$$45R_1 - 42R_2 - 268.125 = 0$$

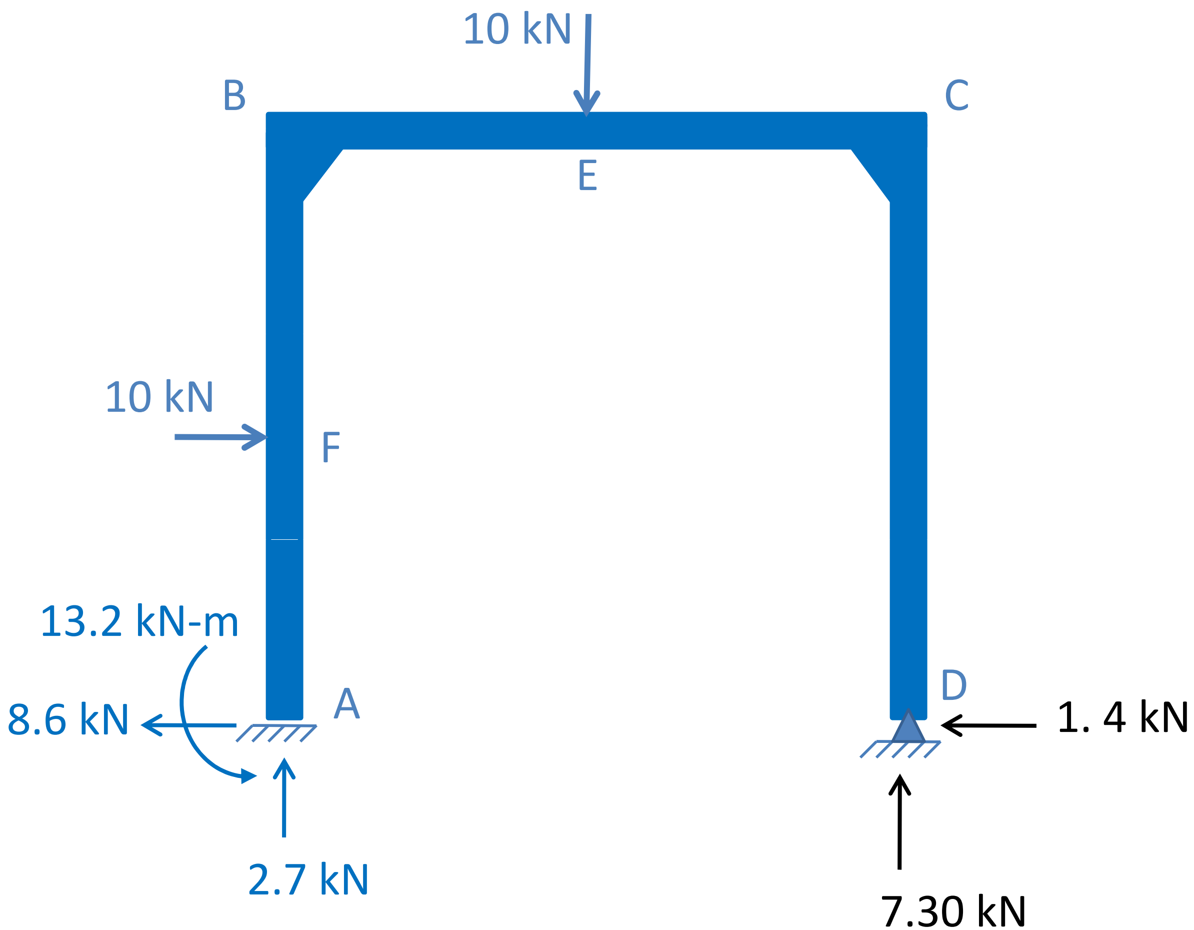
$$-42R_1 + 90.67R_2 + 178.33 = 0$$

solving simultaneously, we have

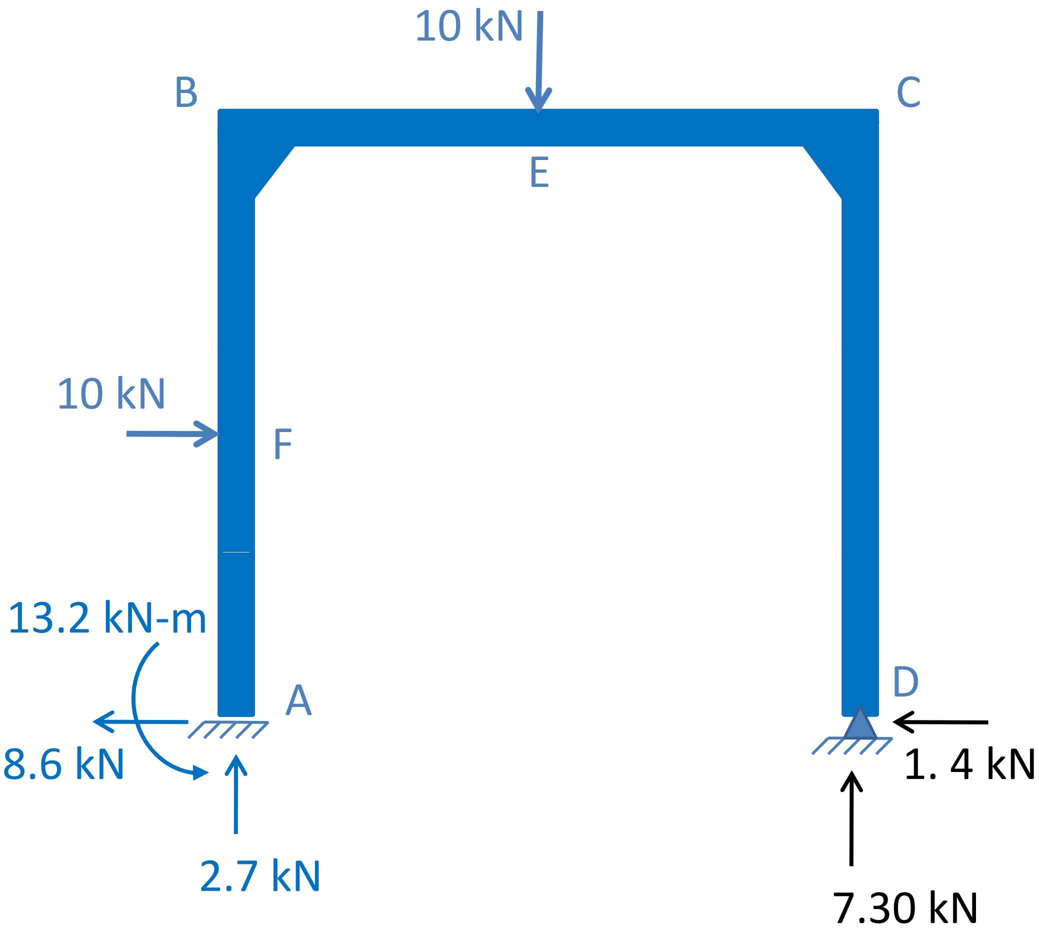
$$R_1 = 7.2608 \cong 7.30\ kN$$

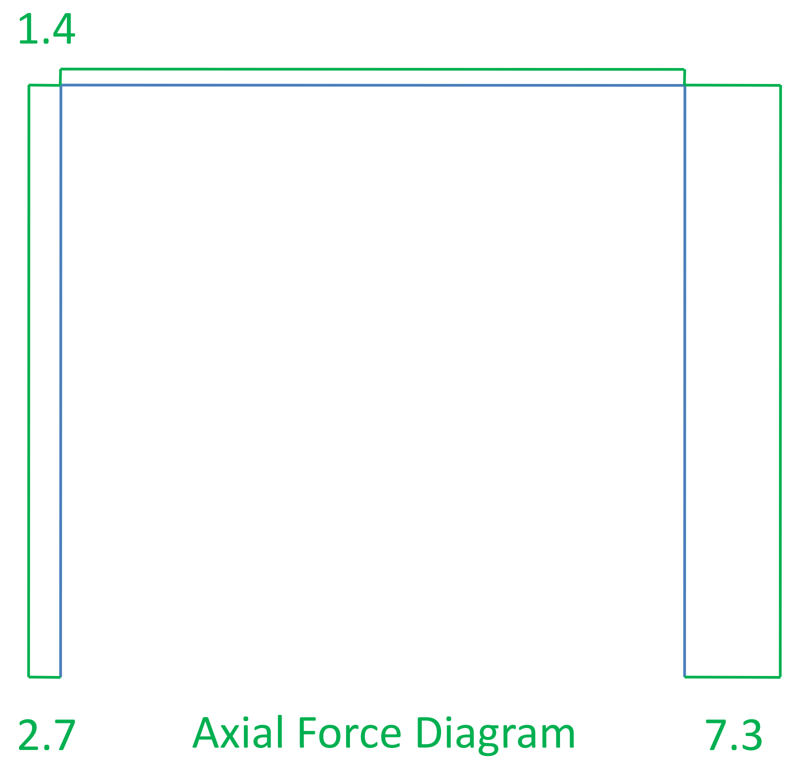
$$R_2 = 1.3964 \cong 1.4\ kN$$

Solution



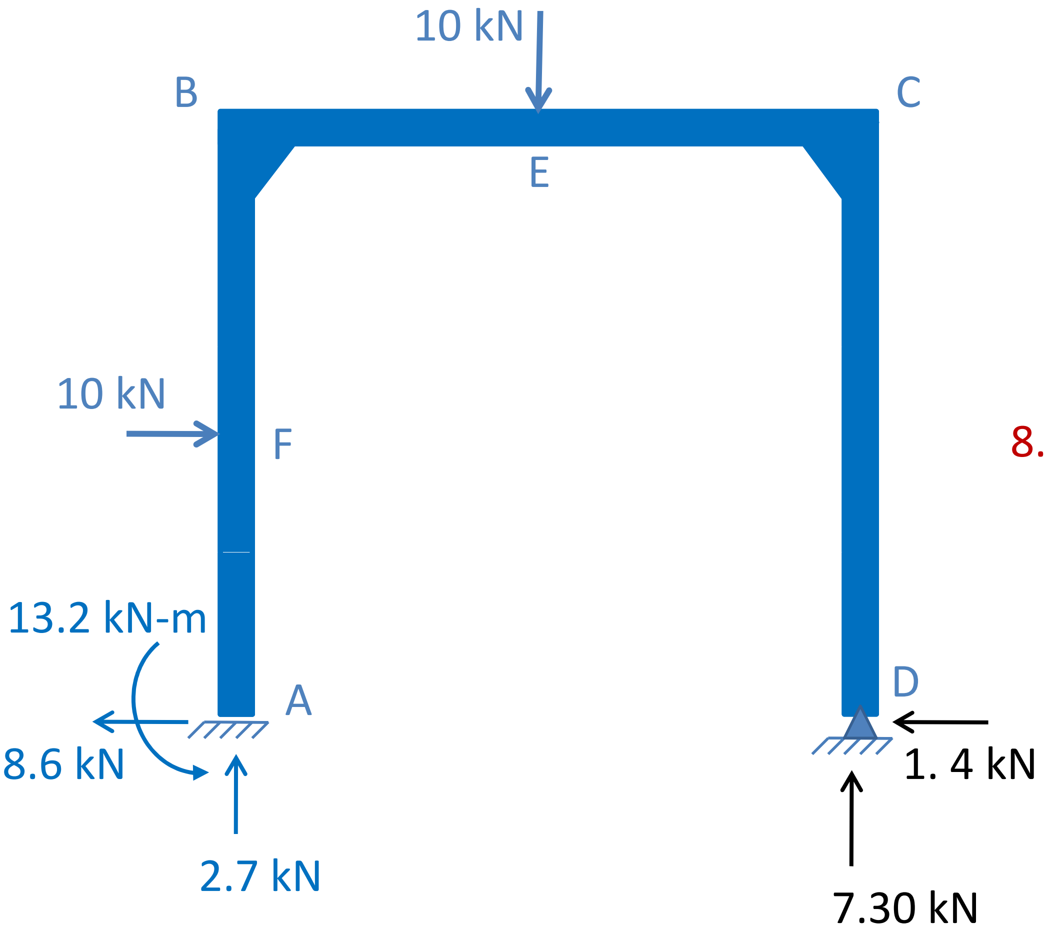
Solution

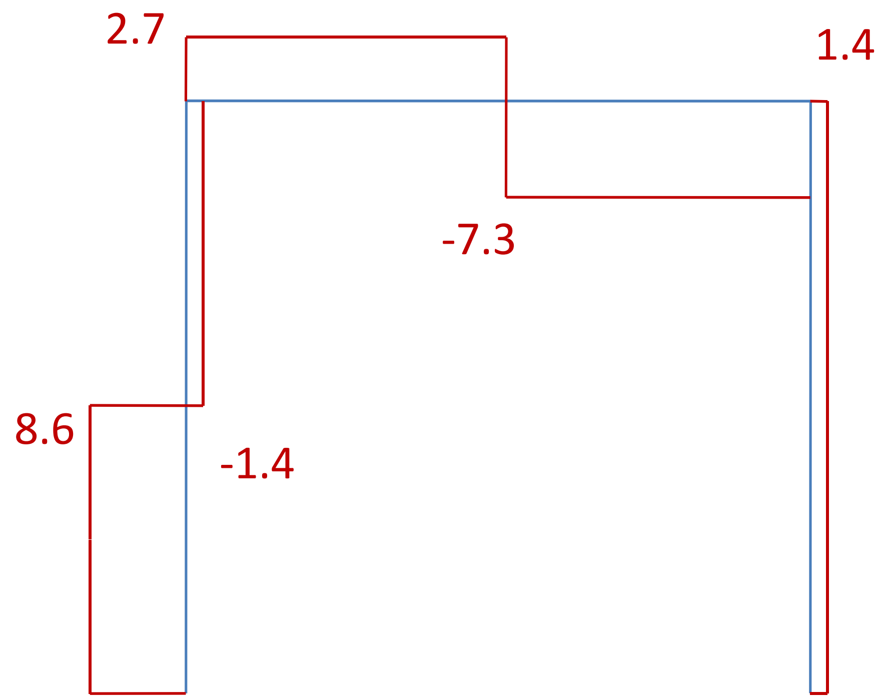




Axial Force Diagram

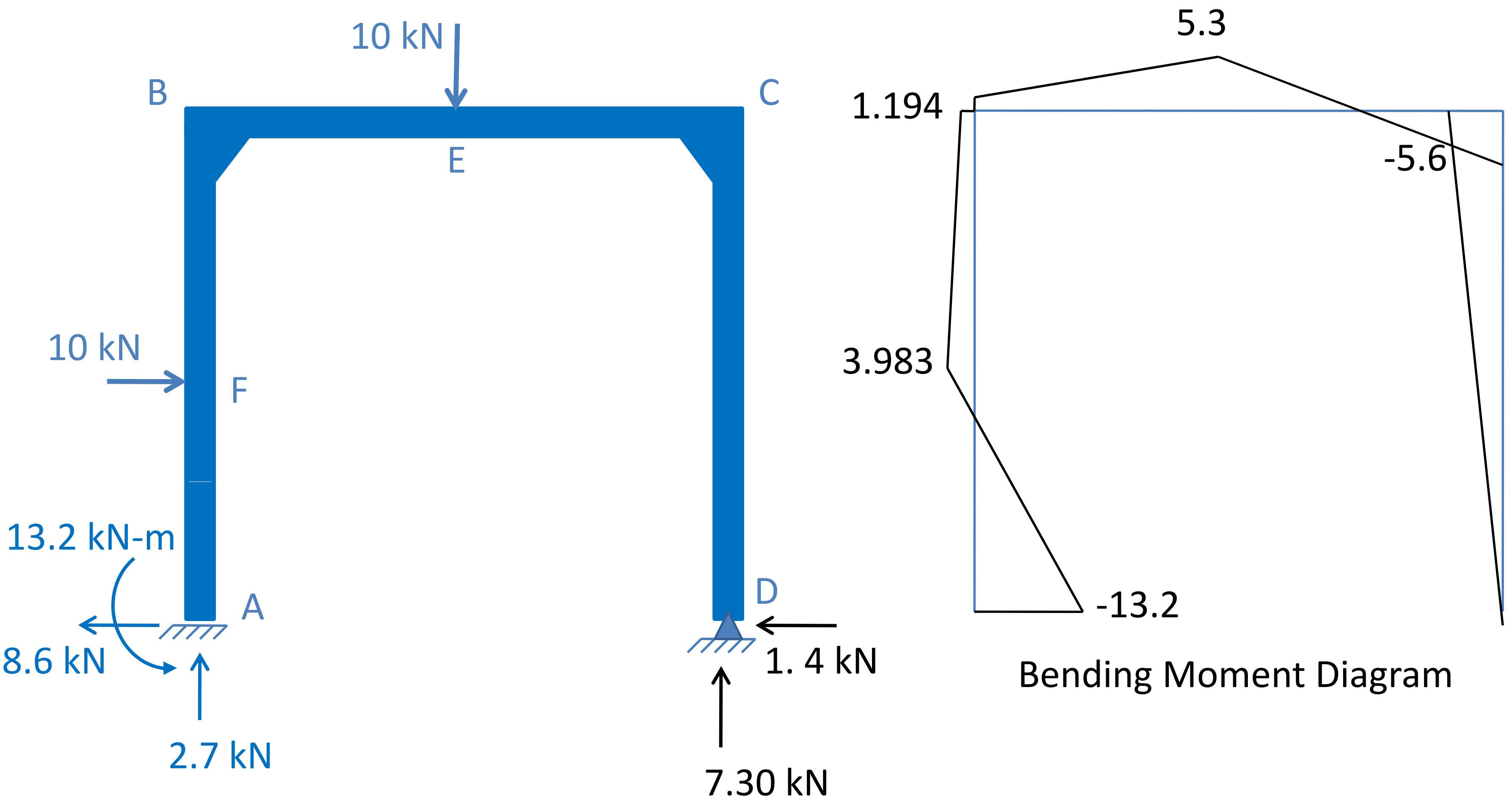
Solution





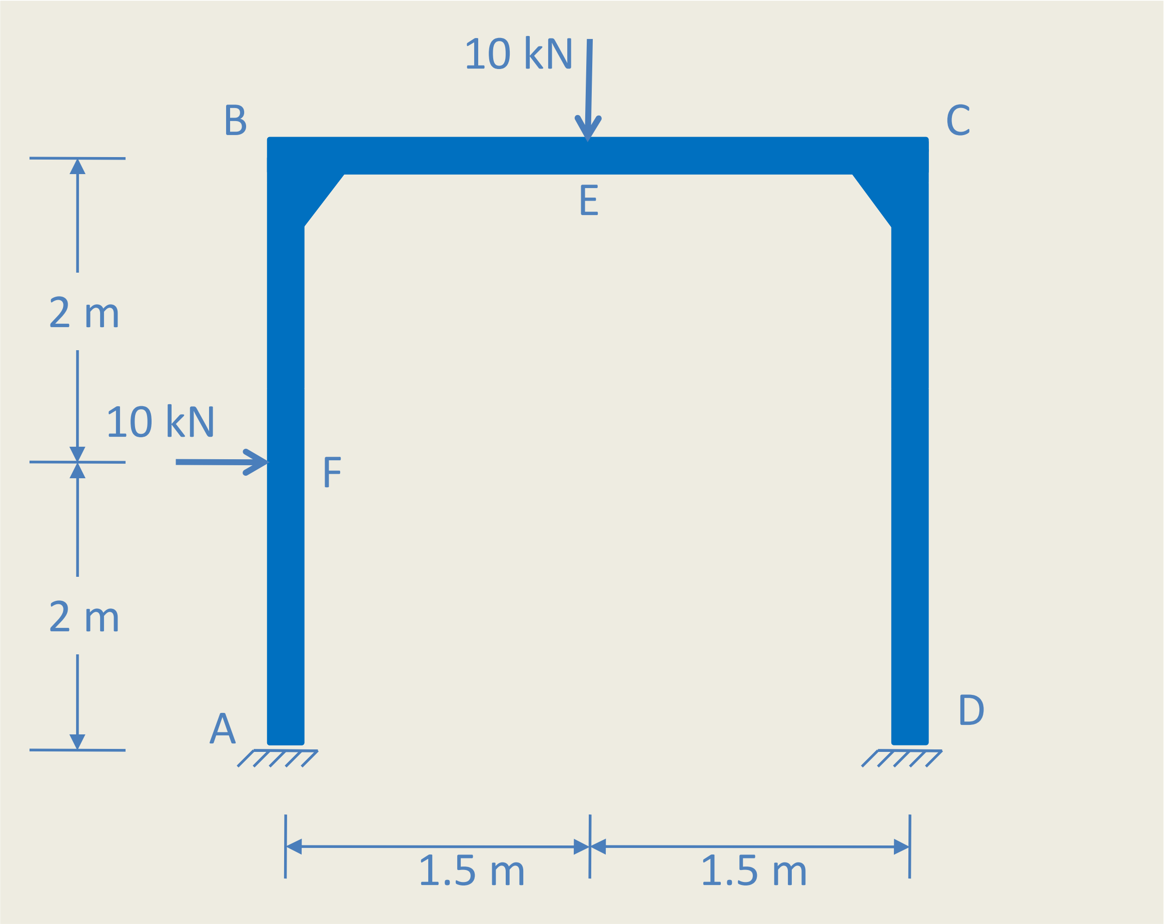
Shear Force Diagram

Solution


10 kN
B
C
E
10 kN
F
13.2 kN-m
A
8.6 kN
2.7 kN
D
1. 4 kN
7.30 kN
5.3
1.194
-5.6
3.983
-13.2
Bending Moment Diagram

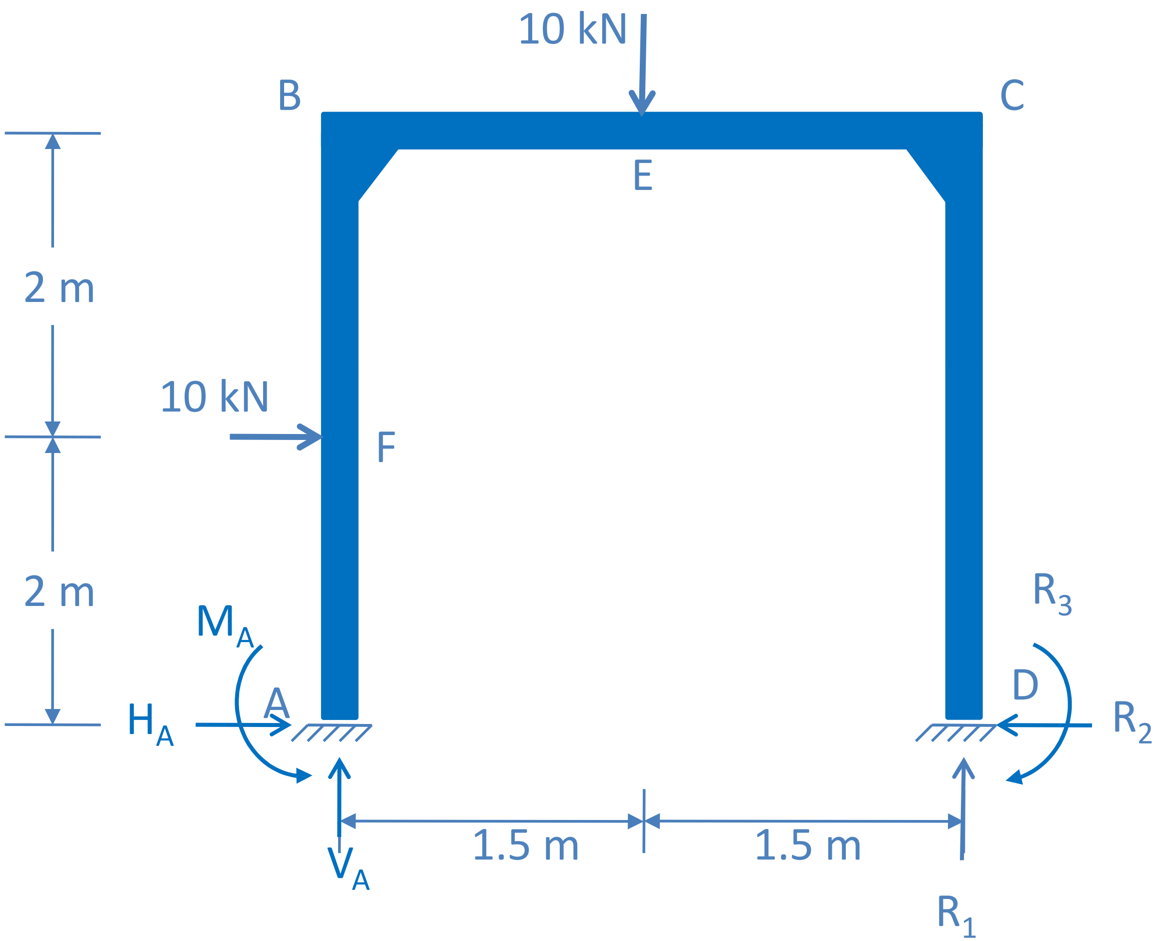
Example 7

Determine the reactions for the frame shown in Fig., by the method of least work. *EI* is constant.



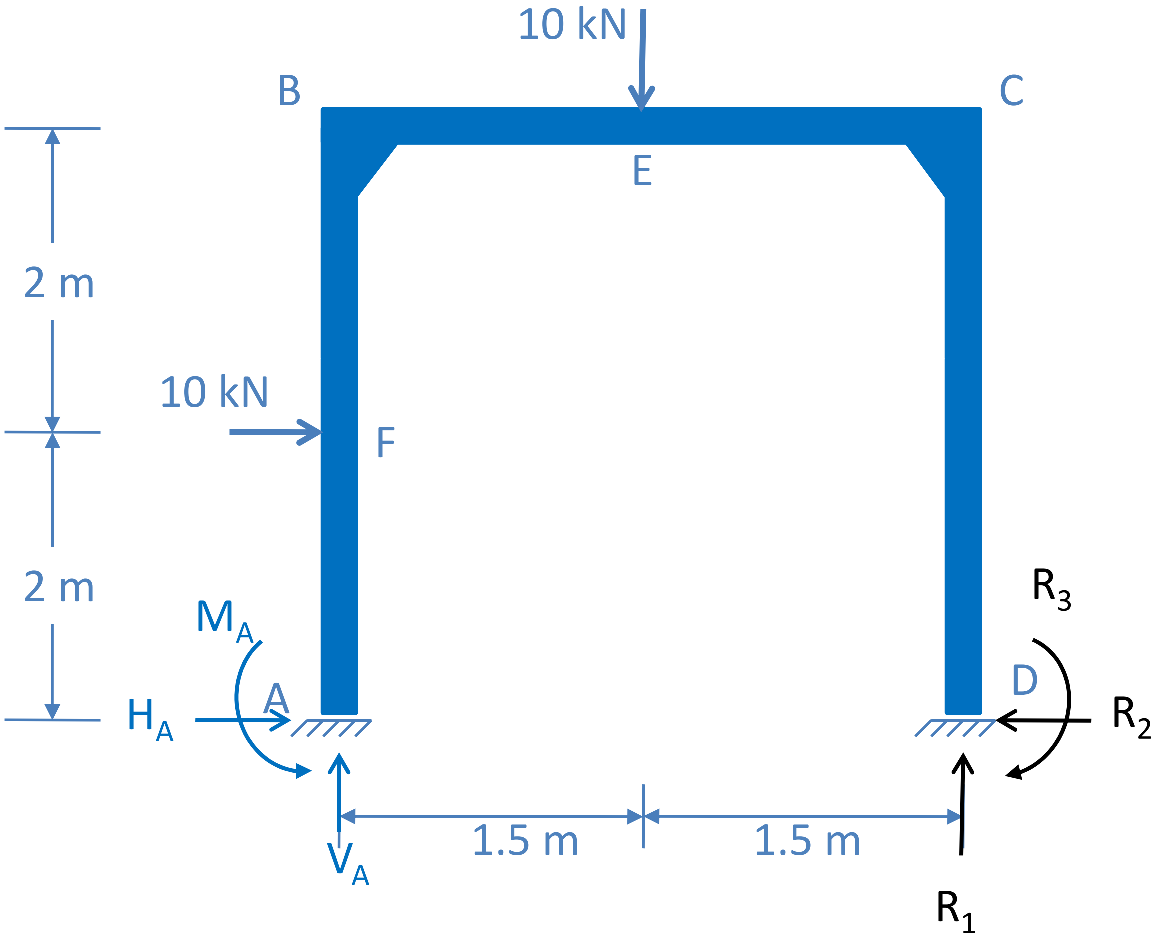
Solution

The structure is determinate to the third degree. It has three redundant reactions.



Solution

Let us choose R_1, R_2, R_3, the reaction at D, to be the redundant.



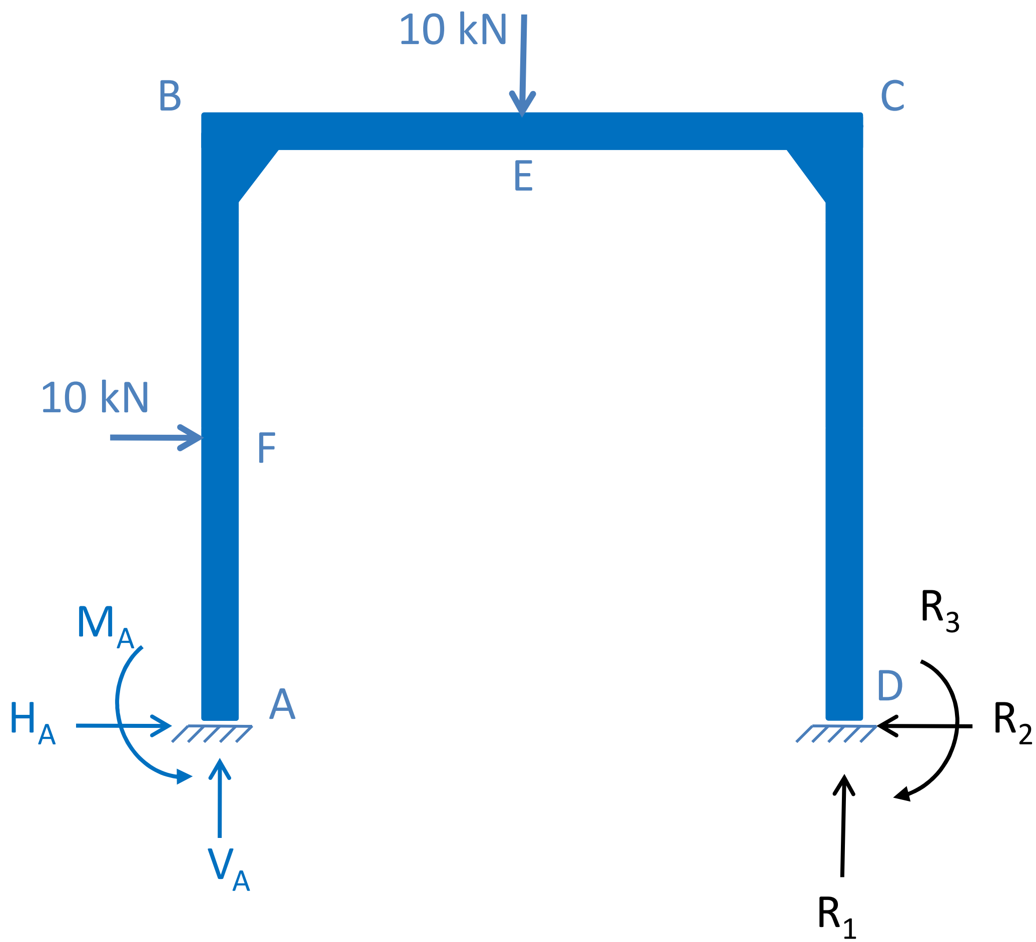
Solution

According to the Principle of Least Work

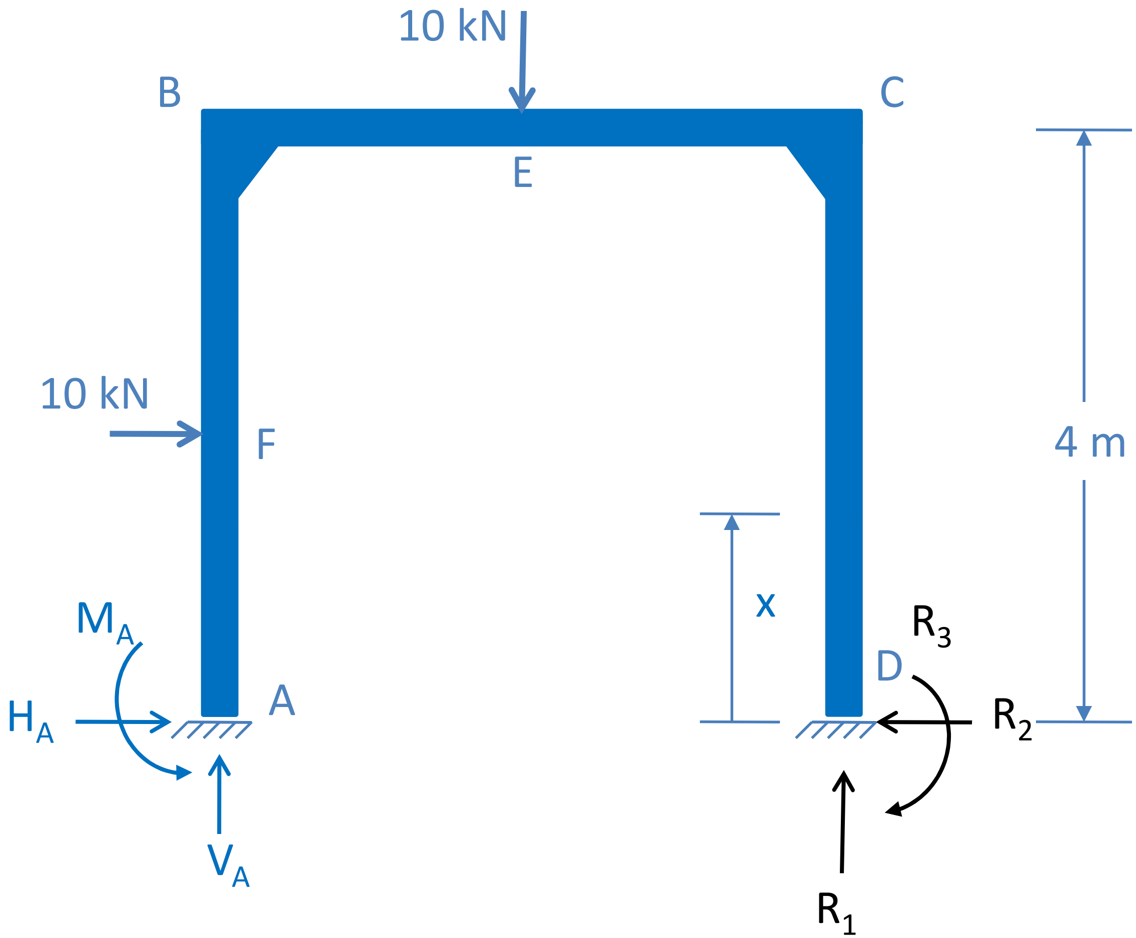
$$\frac{\partial U}{\partial R_1} = \int_0^L \frac{\partial M}{\partial R_1} \frac{M}{EI} dx = 0$$

$$\frac{\partial U}{\partial R_2} = \int_0^L \frac{\partial M}{\partial R_2} \frac{M}{EI} dx = 0$$

$$\frac{\partial U}{\partial R_3} = \int_0^L \frac{\partial M}{\partial R_3} \frac{M}{EI} dx = 0$$

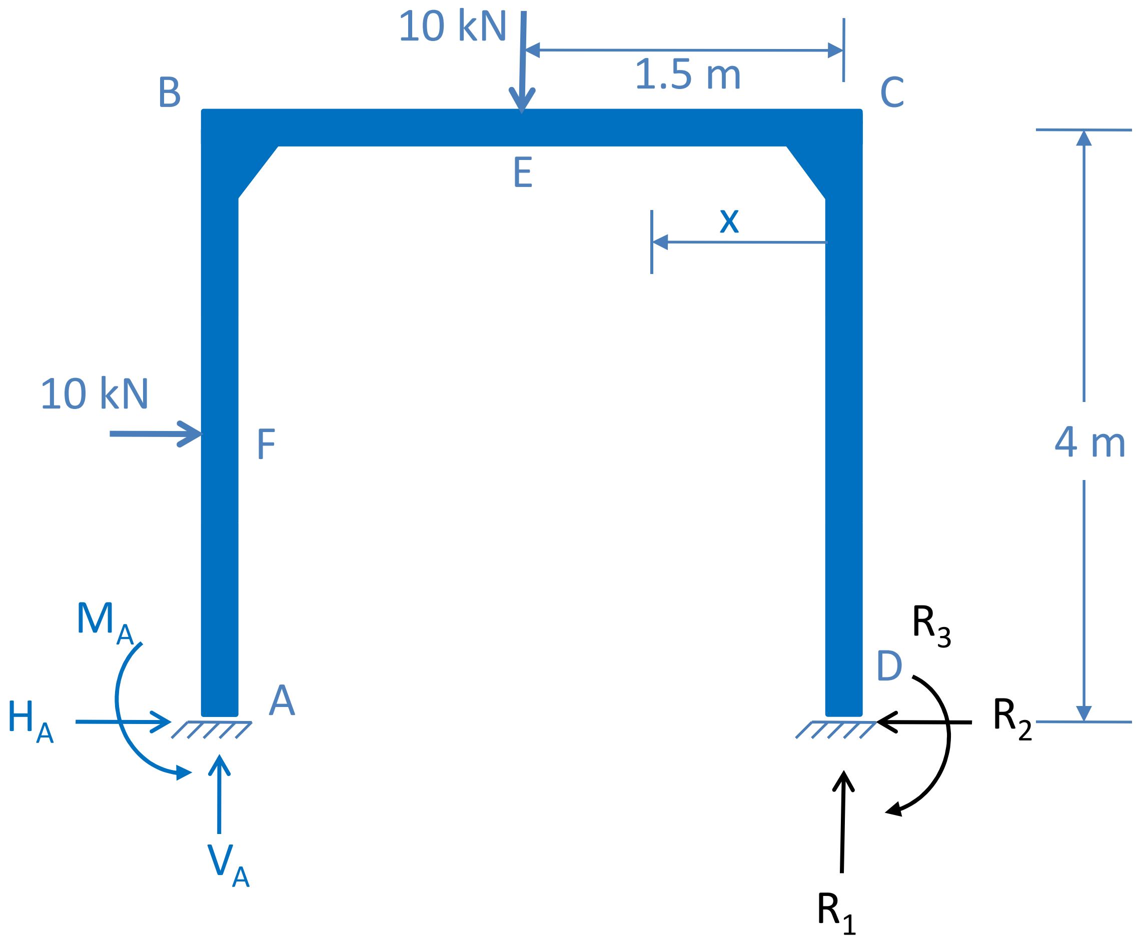


Solution



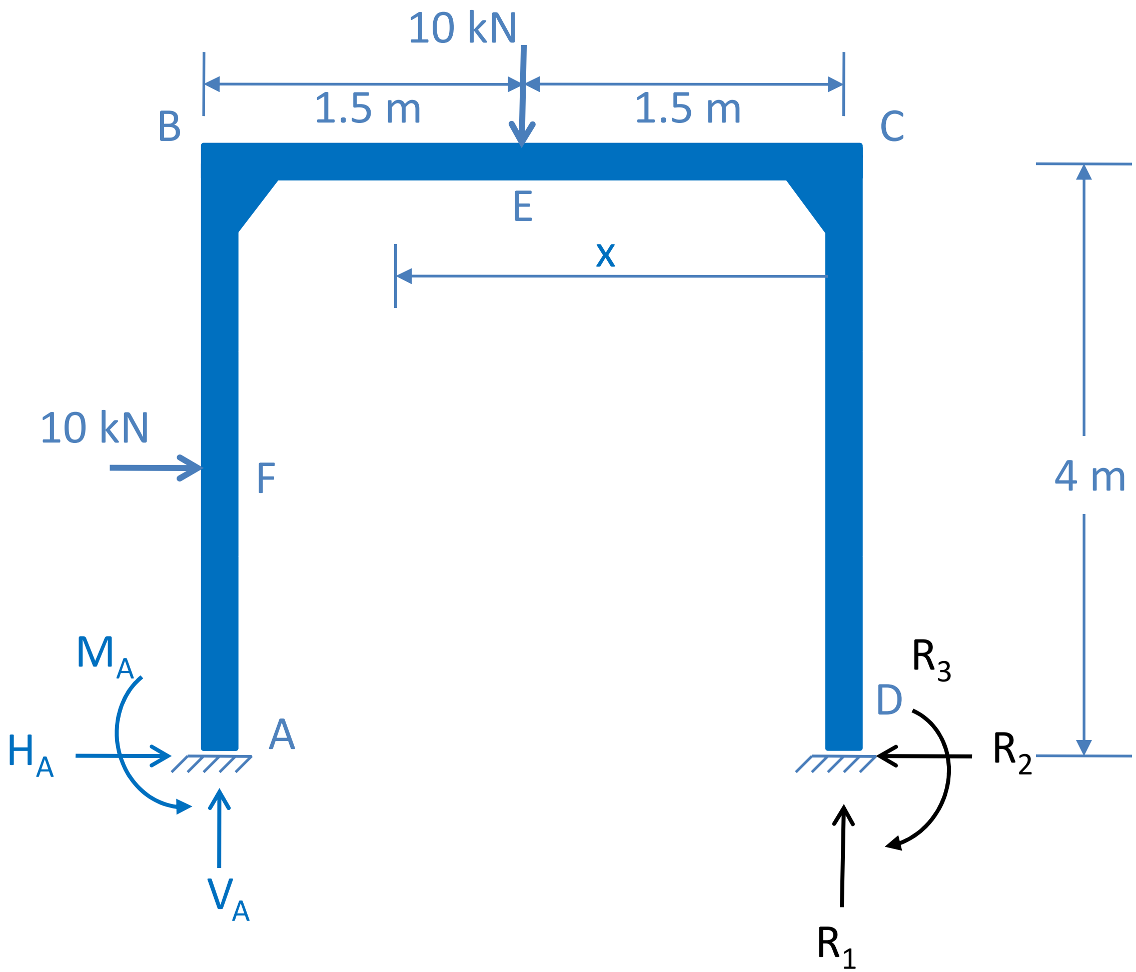
Segment	Origin	Limits	*M*	*∂M/∂R₁*	*∂M/∂R₂*	*∂M/∂R₃*
DC	D	0 – 4	$-R_2.x - R_3$	0	-x	-1

Solution



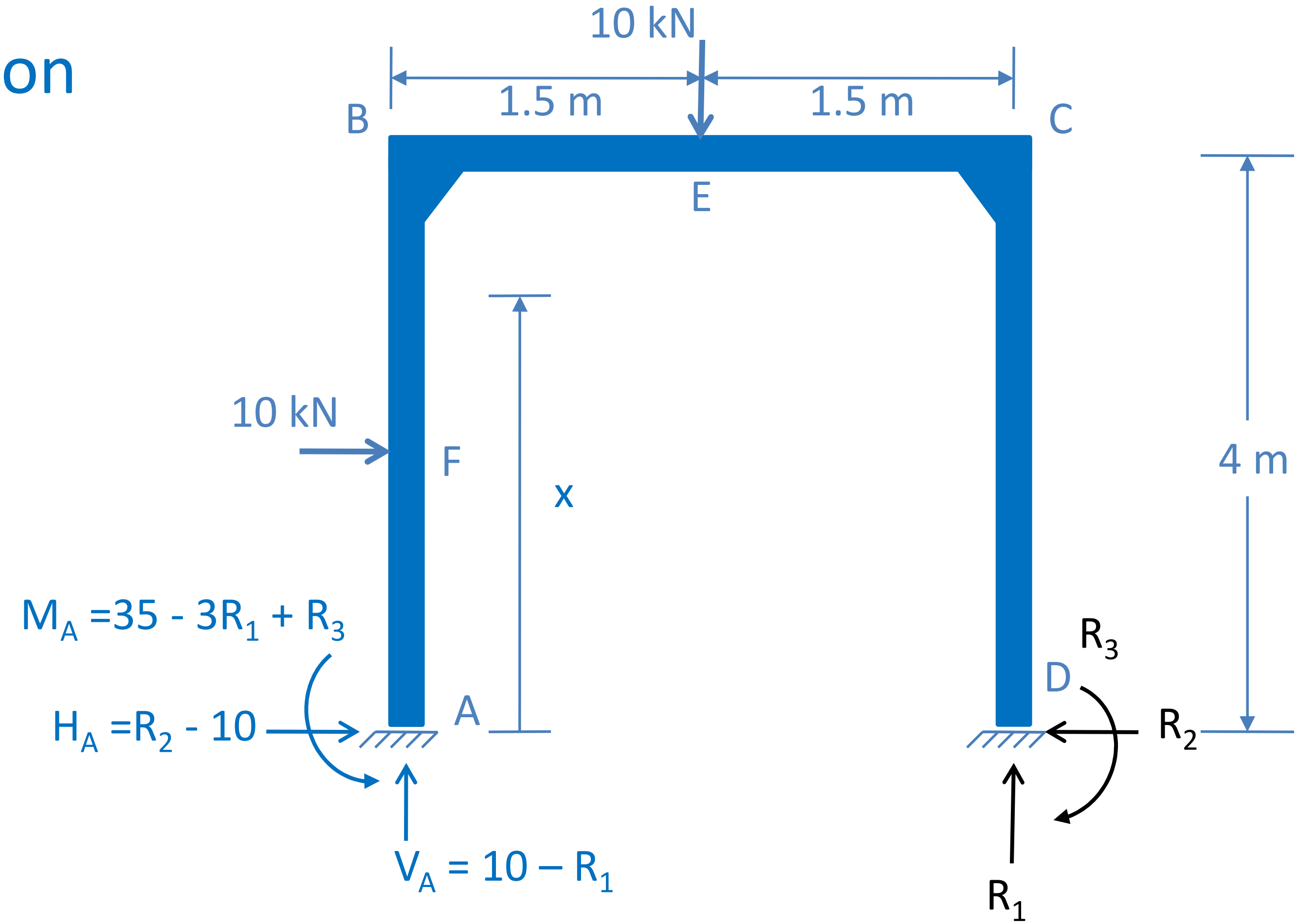
Segment	Origin	Limits	*M*	$\partial M/\partial R_1$	$\partial M/\partial R_2$	$\partial M/\partial R_3$
DC	D	0 – 4	$-R_2.x - R_3$	0	-x	-1
CE	C	0 – 1.5	$R_1.x - R_2.4 - R_3$	x	-4	-1

Solution



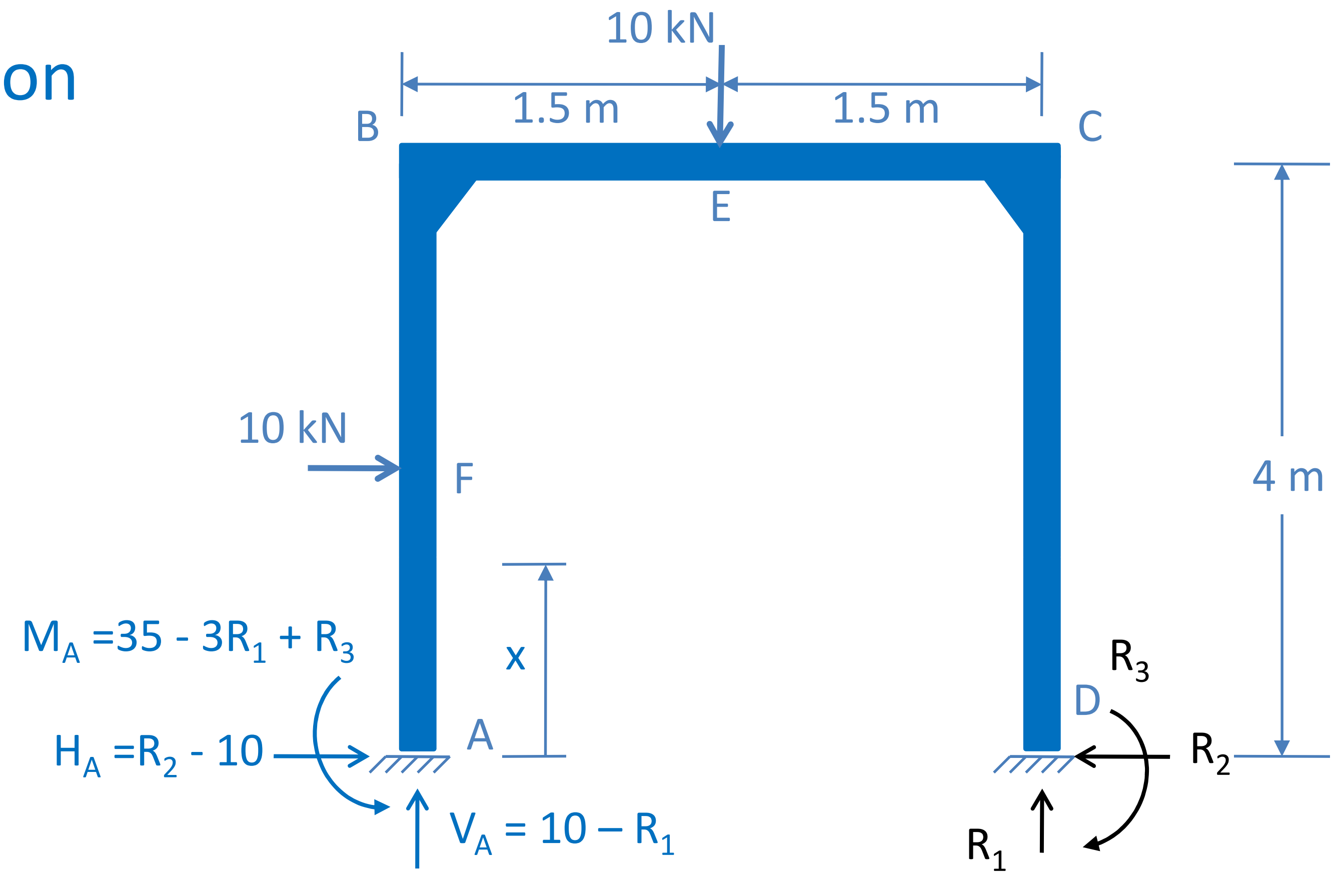
Segment	Origin	Limits	*M*	*∂M/∂R₁*	*∂M/∂R₂*	*∂M/∂R₃*
DC	D	0 – 4	$-R_2.x - R_3$	0	-x	-1
CE	C	0 – 1.5	$R_1.x - R_2.4 - R_3$	x	-4	-1
EB	C	1.5 – 3.0	$R_1.x - R_2.4 - R_3 - 10(x - 1.5)$	x	-4	-1

Solution



Segment	Origin	Limits	*M*	$\partial M/\partial R_1$	$\partial M/\partial R_2$	$\partial M/\partial R_3$
DC	D	0 – 4	$-R_2.x - R_3$	0	-x	-1
CE	C	0 – 1.5	$R_1.x - R_2.4 - R_3$	x	-4	-1
EB	C	1.5 – 3.0	$R_1.x - R_2.4 - R_3 - 10(x - 1.5)$	x	-4	-1
FB	A	2 – 4	$-(R_2 - 10)x - (+35 - 3R_1 + R_3) - 10(x - 2)$	3	-x	-1

Solution



Segment	Origin	Limits	M	$\partial M/\partial R_1$	$\partial M/\partial R_2$	$\partial M/\partial R_3$
DC	D	0 – 4	$-R_2.x - R_3$	0	-x	-1
CE	C	0 – 1.5	$R_1.x - R_2.4 - R_3$	x	-4	-1
EB	C	1.5 – 3.0	$R_1.x - R_2.4 - R_3 - 10(x - 1.5)$	x	-4	-1
FB	A	2 – 4	$-(R_2 - 10)x - (+35 - 3R_1 + R_3) - 10(x - 2)$	3	-x	-1
AF	A	0 – 2	$-(R_2 - 10)x - (+35 - 3R_1 + R_3)$	3	-x	-1

Solution

Segment	Origin	Limits	*M*	$\partial M/\partial R_1$	$\partial M/\partial R_2$	$\partial M/\partial R_3$
DC	D	0 – 4	$-R_2.x - R_3$	0	-x	-1
CE	C	0 – 1.5	$R_1.x - R_2.4 - R_3$	x	-4	-1
EB	C	1.5 – 3.0	$R_1.x - R_2.4 - R_3 - 10(x - 1.5)$	x	-4	-1
FB	A	2 – 4	$-(R_2 - 10)x - (+35 - 3R_1 + R_3) - 10(x - 2)$	3	-x	-1
AF	A	0 – 2	$-(R_2 - 10)x - (+35 - 3R_1 + R_3)$	3	-x	-1

$$\frac{\partial U}{\partial R_1} = \int_0^L \frac{\partial M}{\partial R_1}\frac{M}{EI}dx = 0$$

$$\frac{1}{EI}\int_0^{1.5}(R_1x - R_2 4 - R_3)xdx + \frac{1}{EI}\int_{1.5}^{3.0}[R_1x - R_2 4 - R_3 - 10(x-1.5)]xdx$$

$$+\frac{1}{EI}\int_2^4((-R_2+10)x - (+35 - 3R_1 + R_3) - 10(x-2))3dx + \frac{1}{EI}\int_0^2((-R_2+10)x - 35 + 3R_1 - R_3)3dx = 0$$

$$45R_1 - 42R_2 - 16.5R_3 - 268.125 = 0 \qquad (1)$$

Solution

Segment	Origin	Limits	*M*	$\partial M/\partial R_1$	$\partial M/\partial R_2$	$\partial M/\partial R_3$
DC	D	0 – 4	$-R_2.x - R_3$	0	-x	-1
CE	C	0 – 1.5	$R_1.x - R_2.4 - R_3$	x	-4	-1
EB	C	1.5 – 3.0	$R_1.x - R_2.4 - R_3 - 10(x - 1.5)$	x	-4	-1
FB	A	2 – 4	$-(R_2 - 10)x - (+35 - 3R_1 + R_3) - 10(x - 2)$	3	-x	-1
AF	A	0 – 2	$-(R_2 - 10)x - (+35 - 3R_1 + R_3)$	3	-x	-1

$$\frac{\partial U}{\partial R_2} = \int_0^L \frac{\partial M}{\partial R_2}\frac{M}{EI}dx = 0$$

$$\frac{1}{EI}\int_0^4 (-R_2x - R_3)(-x)dx + \frac{1}{EI}\int_0^{1.5}(R_1x - R_2 4 - R_3)(-4)dx + \frac{1}{EI}\int_{1.5}^{3.0}[R_1x - R_2 4 - R_3 - 10(x - 1.5)](-4)dx$$

$$+\frac{1}{EI}\int_2^4((-R_2 + 10)x - (+35 - 3R_1 + R_3) - 10(x - 2))(-x)dx + \frac{1}{EI}\int_0^2((-R_2 + 10)x - 35 + 3R_1 - R_3)(-x)dx = 0$$

$$-42R_1 + 90.67R_2 + 28R_3 + 178.33 = 0 \qquad (2)$$

Solution

Segment	Origin	Limits	*M*	∂M/∂R₁	∂M/∂R₂	∂M/∂R₃
DC	D	0 – 4	$-R_2.x - R_3$	0	-x	-1
CE	C	0 – 1.5	$R_1.x - R_2.4 - R_3$	x	-4	-1
EB	C	1.5 – 3.0	$R_1.x - R_2.4 - R_3 - 10(x - 1.5)$	x	-4	-1
FB	A	2 – 4	$-(R_2 - 10)x - (+35 - 3R_1 + R_3) - 10(x - 2)$	3	-x	-1
AF	A	0 – 2	$-(R_2 - 10)x - (+35 - 3R_1 + R_3)$	3	-x	-1

$$\frac{\partial U}{\partial R_3} = \int_0^L \frac{\partial M}{\partial R_3}\frac{M}{EI}dx = 0$$

$$\frac{1}{EI}\int_0^4 (-R_2 x - R_3)(-1)dx + \frac{1}{EI}\int_0^{1.5}(R_1 x - R_2 4 - R_3)(-1)dx + \frac{1}{EI}\int_{1.5}^{3.0}[R_1 x - R_2 4 - R_3 - 10(x-1.5)](-1)dx$$
$$+\frac{1}{EI}\int_2^4((-R_2+10)x - (+35 - 3R_1 + R_3) - 10(x-2))(-1)dx + \frac{1}{EI}\int_0^2((-R_2+10)x - 35 + 3R_1 - R_3)(-1)dx = 0$$

$$-16.5R_1 + 28R_2 + 11R_3 + 91.25 = 0 \qquad (3)$$

Solution

$$45R_1 - 42R_2 - 16.5R_3 - 268.125 = 0 \qquad (1)$$

$$-42R_1 + 90.67R_2 + 28R_3 + 178.33 = 0 \qquad (2)$$

$$-16.5R_1 + 28R_2 + 11R_3 + 91.25 = 0 \qquad (3)$$

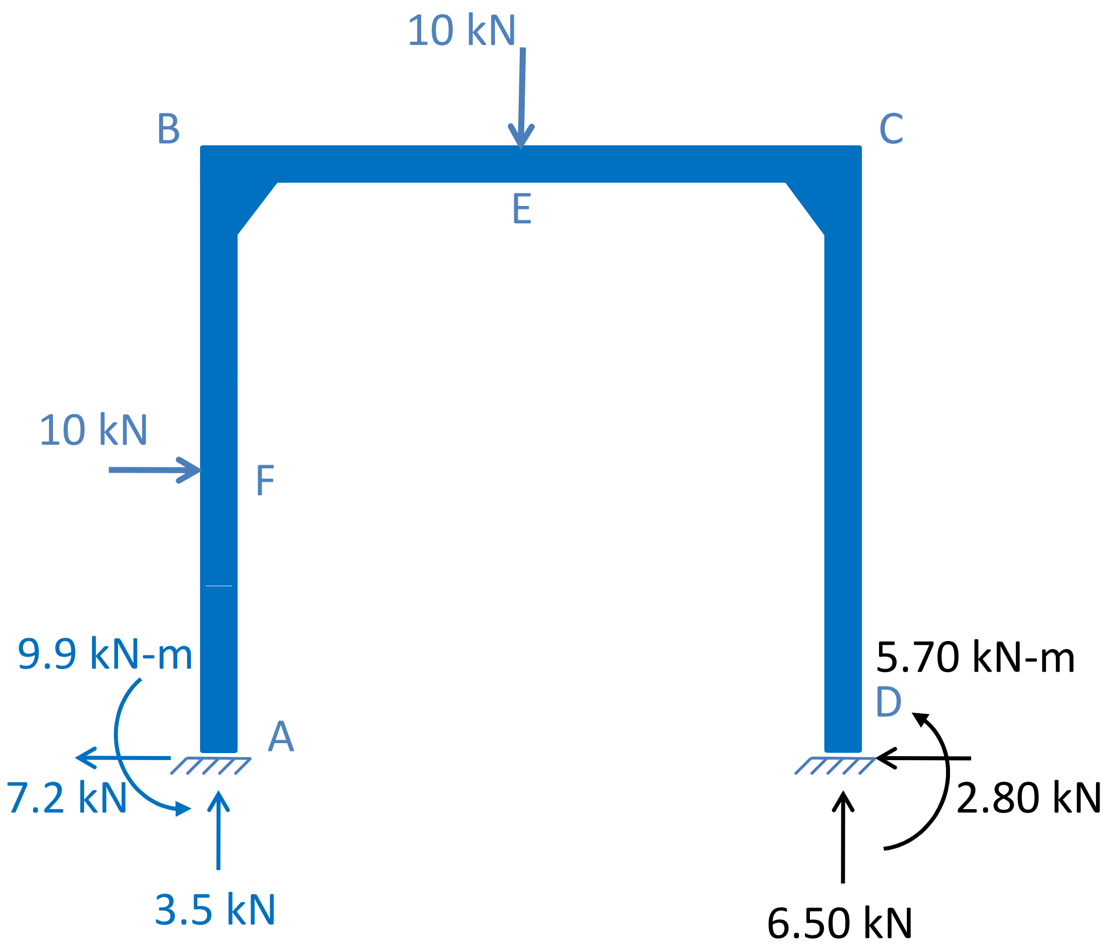
By solving simultaneously, we have

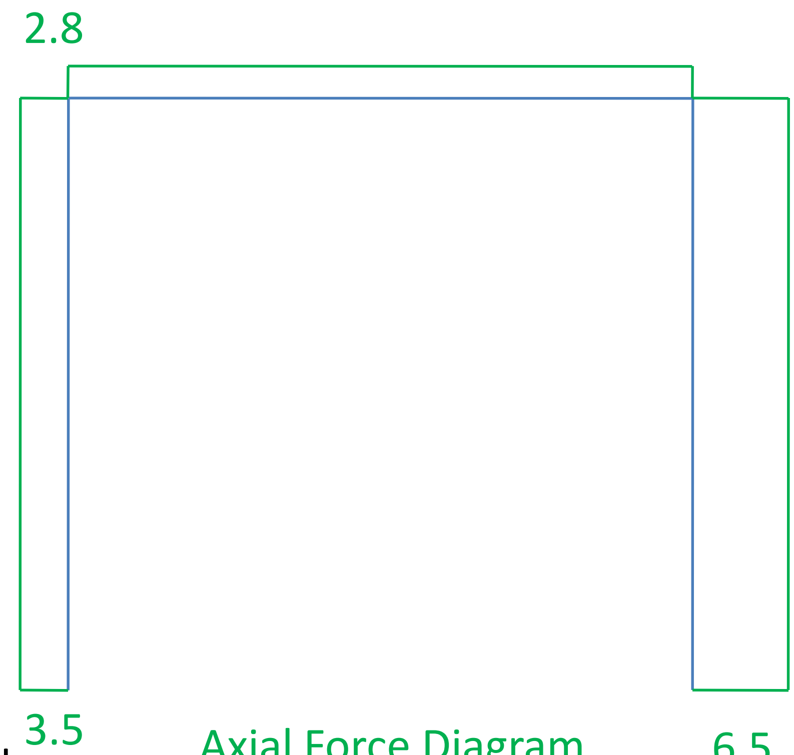
$$R_1 = 6.48\,kN \cong 6.50 \uparrow$$

$$R_2 = 2.77\,kN \cong 2.80 \leftarrow$$

$$R_3 = -5.63\,kN - m \cong 5.70\,kN - m \circlearrowright$$

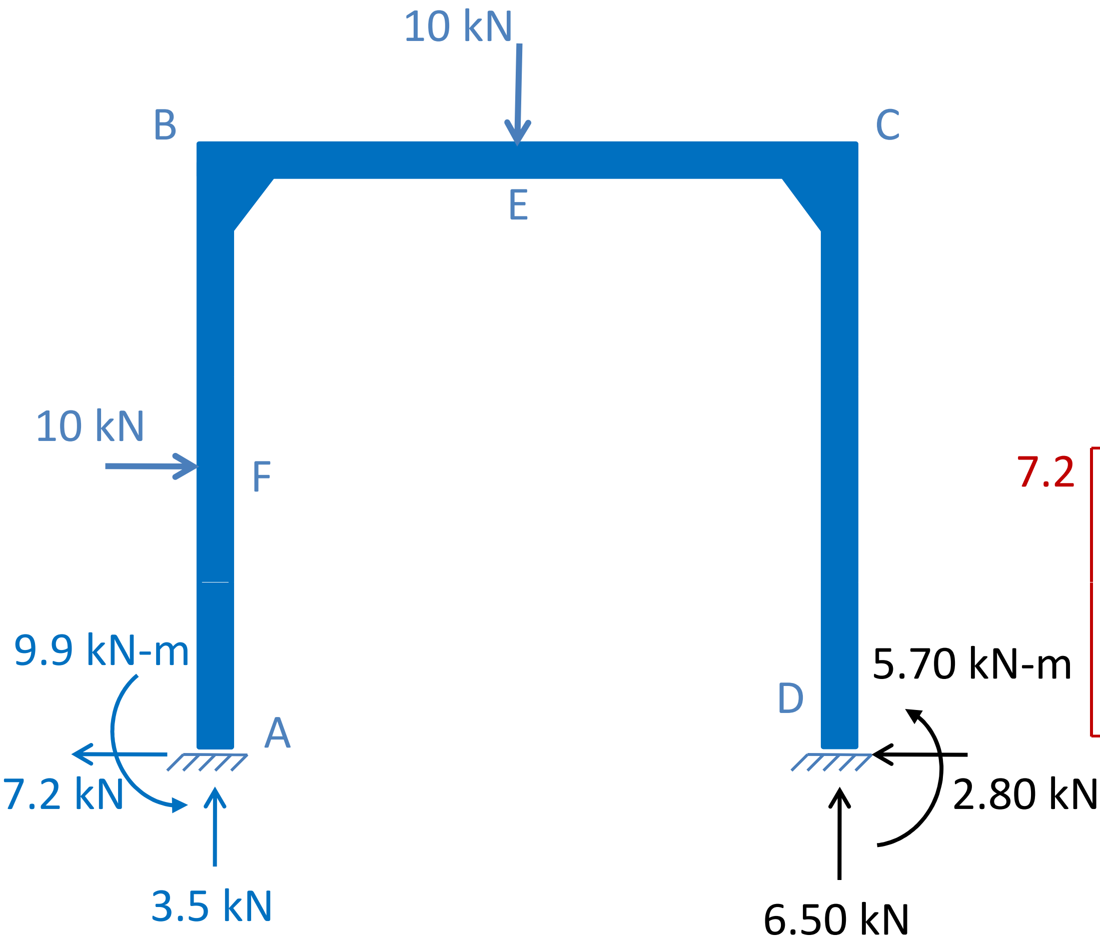
Solution

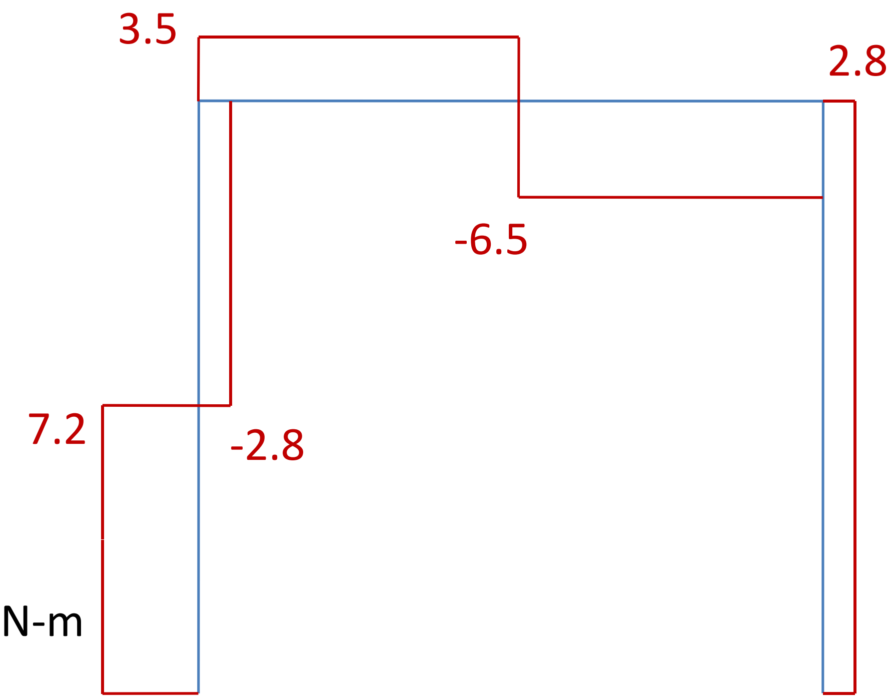




Axial Force Diagram

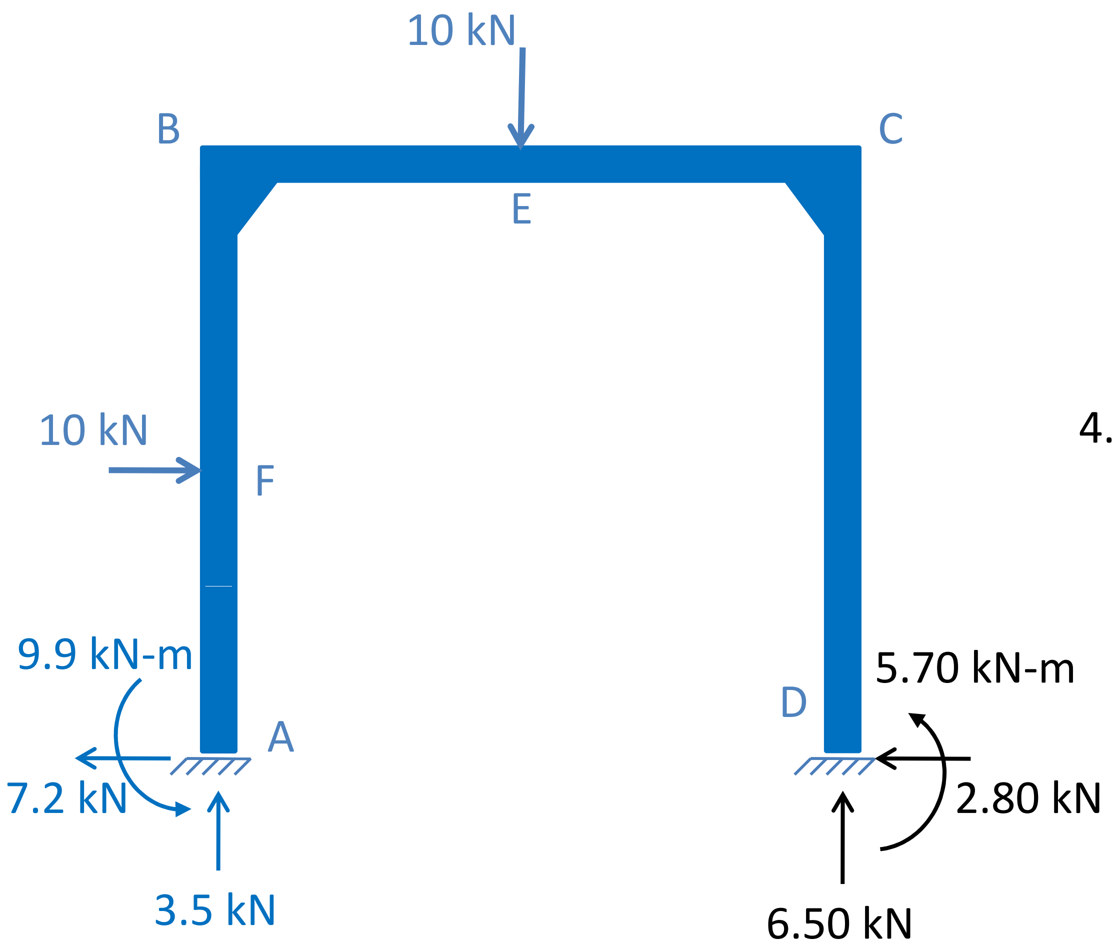
Solution

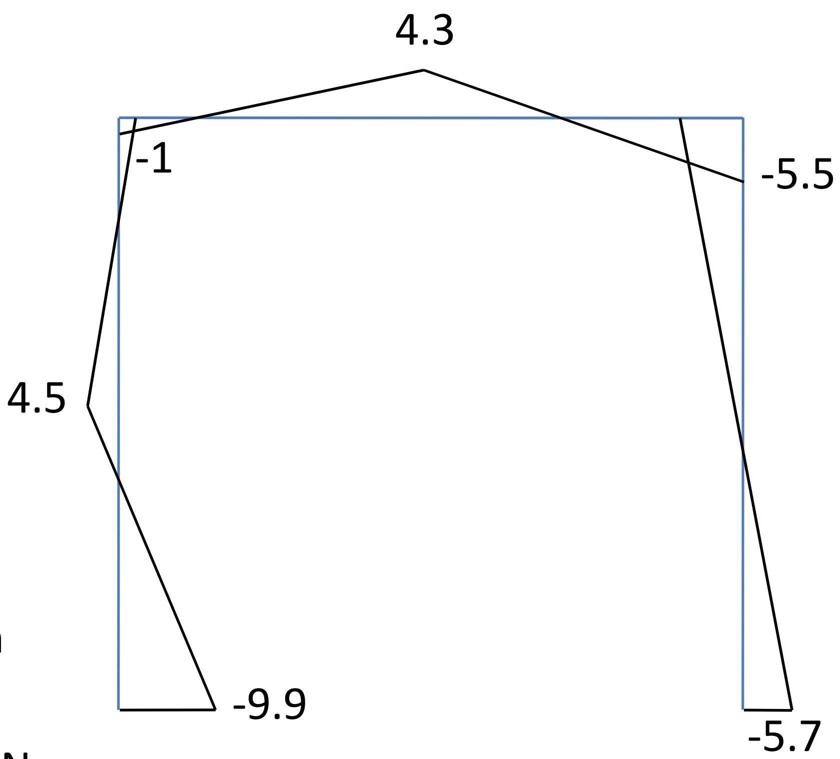




Shear Force Diagram

Solution





Bending Moment Diagram