Plastic Analysis

Theory of Structures-II
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Plastic Analysis

"The plastic method of structural analysis is concerned with determining the maximum loads that a structure can sustain before collapse."

The collapse load is known as failure load, the ultimate load, and the limit load.

A number of comprehensive textbooks dealing with plastic design are available.

The plastic method is applicable to structures constructed with an ideal elastic-plastic material such as mild steel.

If a piece of mild steel is subjected to a tensile force, it will begin to elongate. If tensile force is increased at a constant rate the amount of elongation will increase constantly within certain limits.

In other words, elongation will double when the stress goes from 6000 to 12,000 psi.

When the tensile stress reaches a value roughly equal to one-half of the ultimate tensile strength, the elongation will begin to increase at a greater rate without a corresponding increase in the stress.

The largest stress for which Hook's law applies or the highest point on the straight-line portion of the stress-strain diagram is the "proportional limit".

The largest stress that a material can withstand without being permanently deformed is called the "elastic limit".

This value is seldom actually measured and for most engineering materials including structural steel is synonymous with the proportional limit.

For this reason the term "proportional elastic limit" is sometimes used.

The stress at which there is a decided increase in the elongation or strain without a corresponding increase in stress is said to be the "yield point".

It is the first point on the stress-strain diagram where a tangent to the curve is horizontal.

The strain that occurs before the yield point is referred to as “elastic strain”.

The strain that occurs after the yield point, with no increase in stress is referred to as the “plastic strain”.

The plastic strain usually vary from 10 to 15 times the elastic strains.

Yielding of steel without stress increase may be thought to be a severe disadvantage.

But in actuality, it is a very useful characteristic. It has often performed the wonderful service of preventing failure due to omissions or mistakes on the designer's part.

If the stress at one point in a ductile steel structure reach the yield point, that part of the structure will yield locally without stress increase, thus preventing premature failure.

This ductility allows the stresses in a steel structures to be readjusted.

In other words, very high stresses caused by fabrication, erection, or loading will tend to equalize themselves.

It can be said that a steel structure has a reserve of plastic strain that enables it to resist overloads and sudden shocks.

After the plastic strain there is a range where additional stress is necessary to produce additional strain and this is called “strain hardening”.

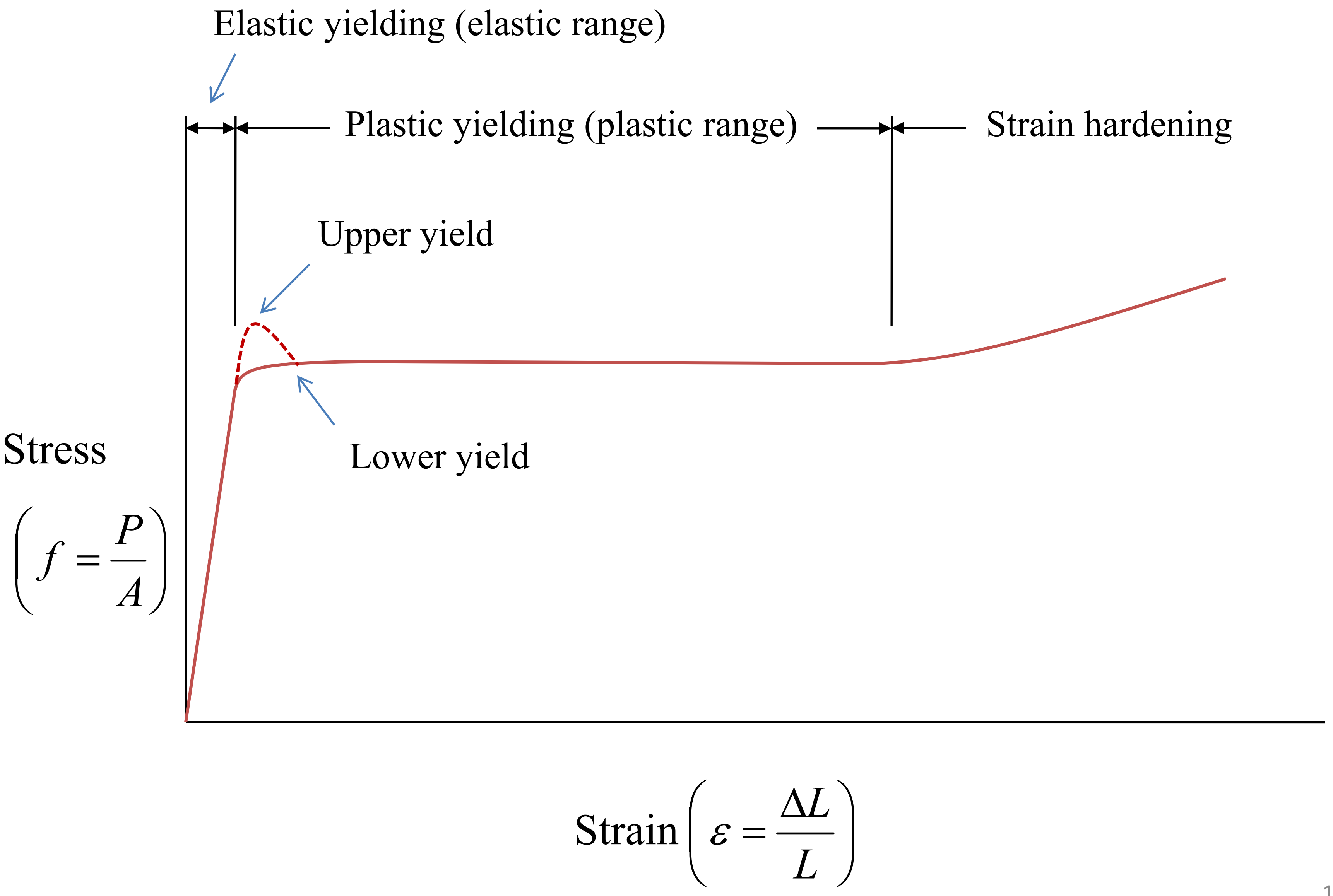
This portion of the diagram is not too important to today's designer.

A familiar stress-strain diagram for mild structural steel is shown in next slide.

Only the initial part of the curve is shown because of the great deformations that occurs before failure.

At failure in the mild steels the total strains are from 150 to 200 times the elastic strains.

Stress-Strain Diagram

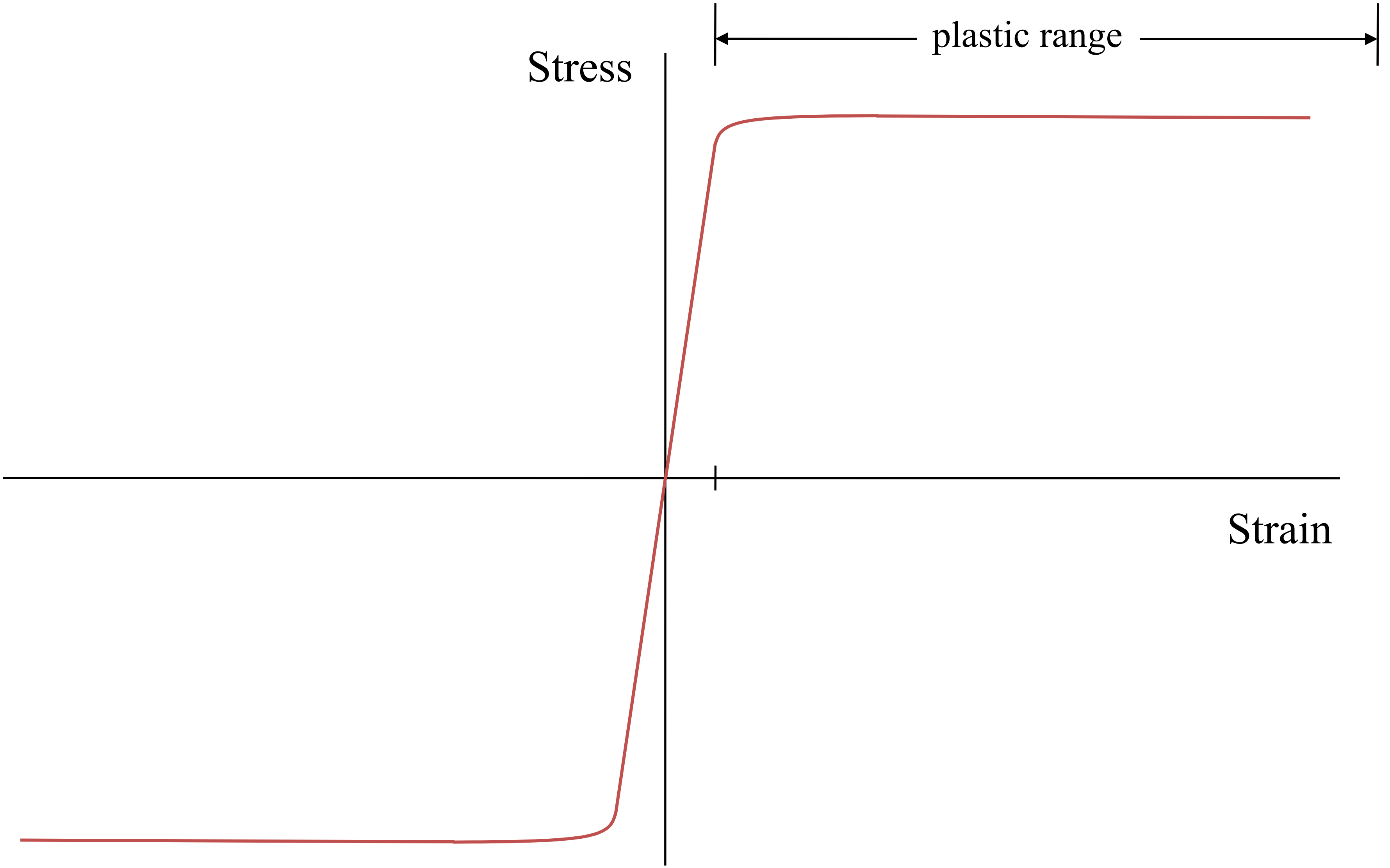


The curve will actually continue up to its maximum stress value and then “tail off” before failure.

A sharp reduction in the cross section of the member takes place (called “necking”) followed by the failure.

This stress-strain curve is of usual ductile structural steel and is assumed to be same for members in tension or compression.

Stress-Strain Diagram



The shape of the diagram varies with the speed of loading, the type of steel, and the temperature.

One such variation is shown in the figure by the dotted line marked "upper yield".

This shape stress-strain curve is the result when a mild steel has the load applied rapidly whereas the "lower yield" is the case for slow loading.

Plastic theory was introduced by J. A. Van den Broek in 1939, which he called "Limit Design".

In this theory, rather than basing designs on the allowable-stress method, the problem is handled by considering the greatest load that can be carried by the structure acting as a unit.

The resulting designs are quite interesting to the structural designer as they offer several advantages.

These include the following:

1. Savings in cost of steel up to 10 – 15%.

2. More accurate estimate of the maximum load a structure can support.
3. For some complicated structures it is easier to apply as compared to elastic analysis.
4. Structures are often subjected to large stresses that are difficult to predict such as those caused by settlement, erection, etc., plastic design provides plastic deformation for such situations.

The method has some following disadvantages:

1. Plastic design is of little value for the high-strength brittle steel.
2. Plastic design today is not satisfactory for situations in which fatigue stresses are a problem.
3. Columns designed by plastic theory provide little savings.
4. Unstable plastic structures are more difficult to detect than are unstable elastic structures.

Theory of Plastic Analysis

The basic theory is that the stress-distribution changes after the stresses at certain points in a structure reach the yield point.

Those parts of the structure that have been stressed to the yield point cannot resist additional stresses.

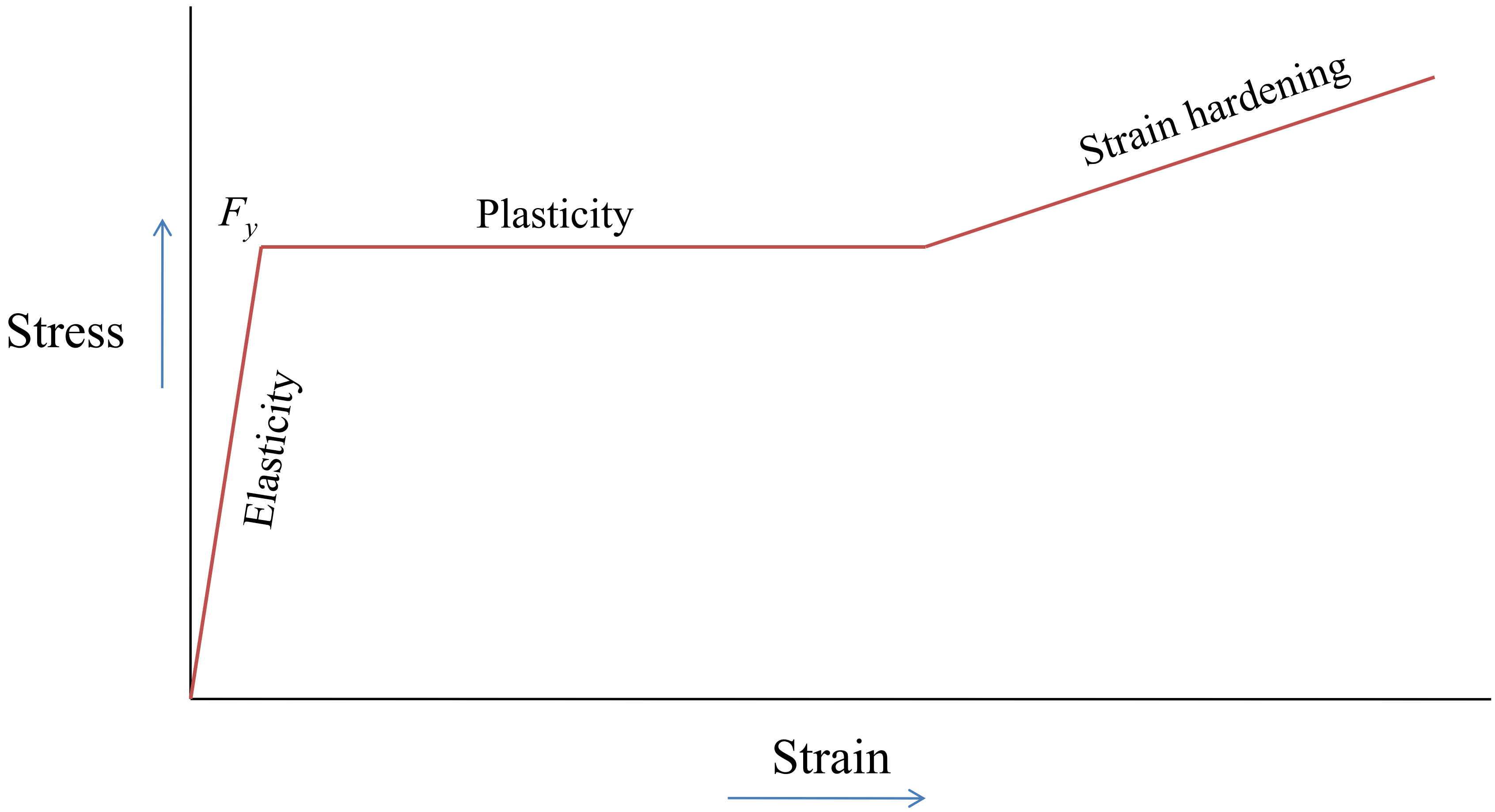
These parts will yield the amount required to permit the load or stresses to be transferred to other parts of the structure.

The stress-strain diagram is assumed to have the idealized shape shown.

The yield point and the proportional limit are assumed to occur at the same point for this steel, and the stress-strain diagram is assumed to be a perfectly straight line in the plastic range.

Beyond the plastic range there is the range of strain hardening.

Idealized Stress-Strain Diagram

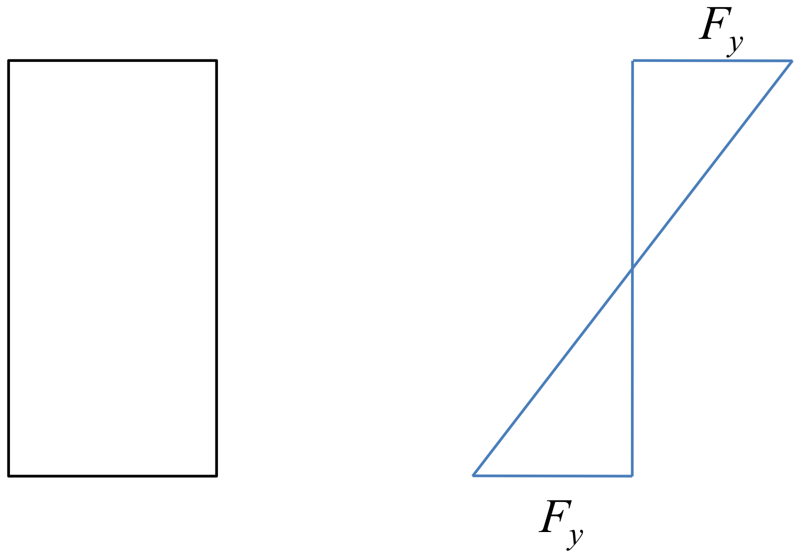


In this range steel members can withstand additional stresses, but from practical point of view the strains and deflections occurring are so large that they cannot be considered.

The Plastic Hinge

As the bending moment is increased at a particular section of a beam there will be a linear variation of stress until the yield stress is reached in the outermost fibers.

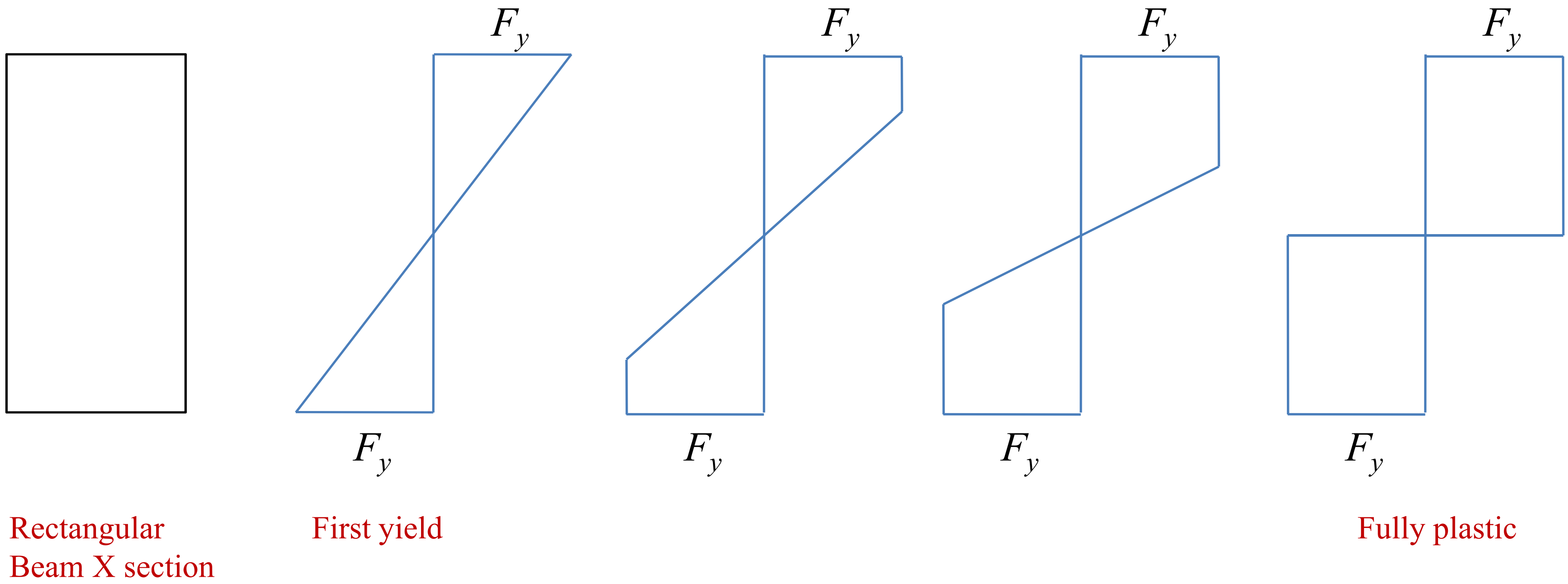
For a rectangular beam the illustration of stress variation in the elastic range is shown in the figure below.



The "yield moment" of a cross section is defined as the moment that will just produce the yield stress in the outermost fiber of the section.

If the moment is increased beyond the yield moment, the outermost fibers will continue to have the same stress but will yield, and the duty of providing the necessary additional resisting moment will fall on the fibers nearer to the neutral axis.

This process will continue with more and more parts of the beam cross section stressed to the yield point until, finally, a fully "plastic distribution" is approached.



When the stress distribution has reached this stage, a plastic hinge is said to have formed because no additional moment can be resisted at the section.

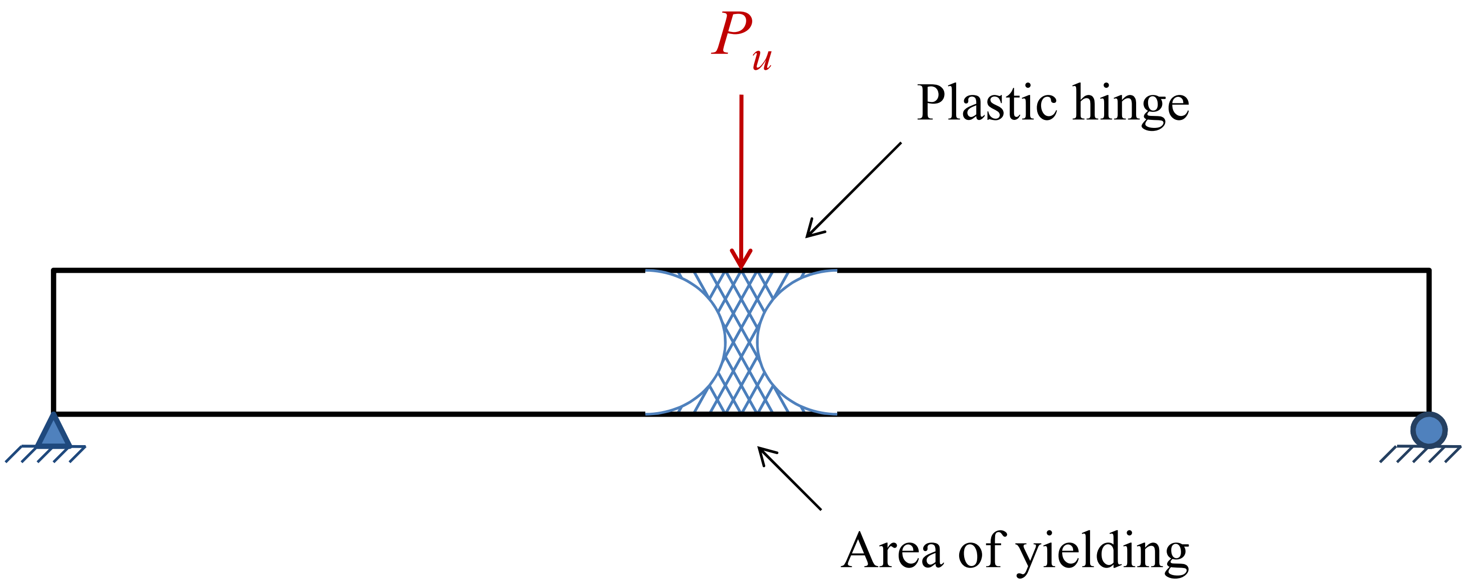
Any additional moment applied at the section will cause the beam to rotate with little increase in stress.

The "plastic moment" is the moment that will produce full plasticity in a member cross section and create a plastic hinge.

The ratio of the plastic moment M_p to the yield moment M_y is called the "shape factor".

$$\text{Shape Factor} = \upsilon = \frac{M_p}{M_y}$$

The shape factor equals 1.50 for rectangular sections and varies from about 1.10 to 1.20 for standard rolled beam sections.



The Yield Moment

The yield moment M_y is equal to the yield stress F_y times the elastic section modulus Z.

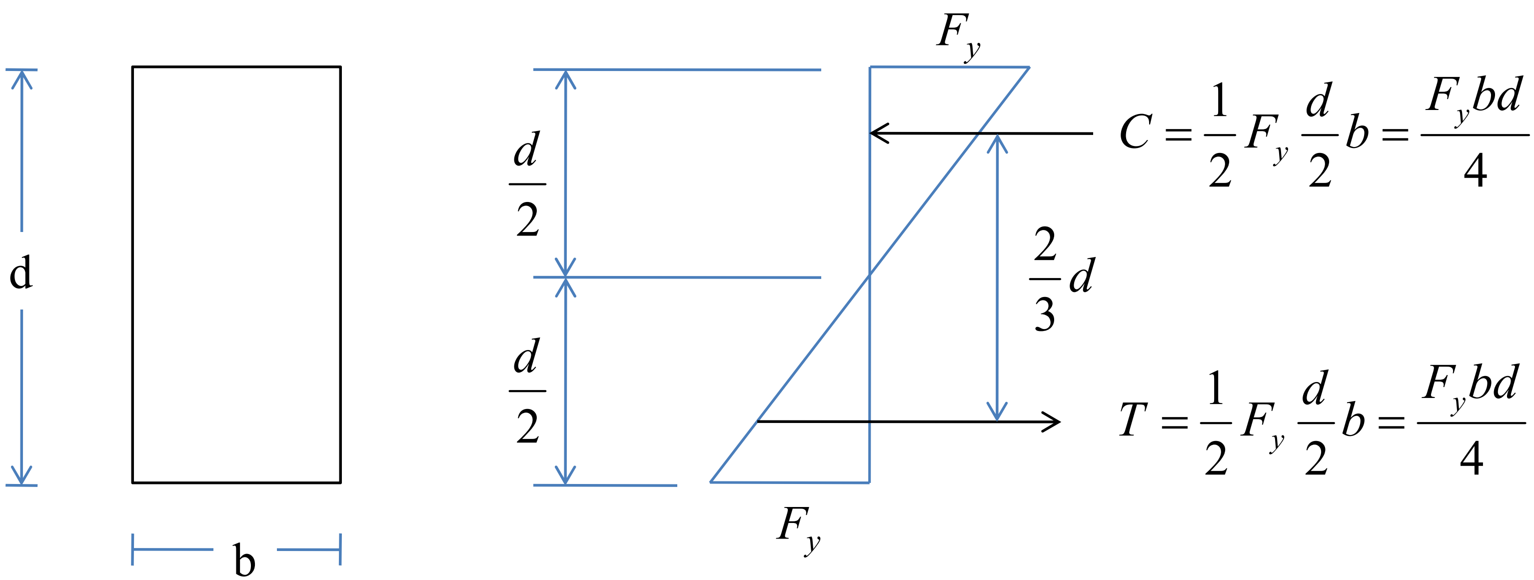
$$M_y = F_y \times Z$$

The elastic section modulus equals I/c or $bd^2/6$ for a rectangular section.

$$M_y = F_y \frac{bd^2}{6}$$

The same value can be obtained by considering the resisting internal couple.

The resisting moment equals T or C times the lever arm between them as follows:



$$M_y = \left(F_y \frac{bd}{4} \right) \left(\frac{2}{3} d \right) = \frac{F_y bd^2}{6}$$

The Plastic Moment

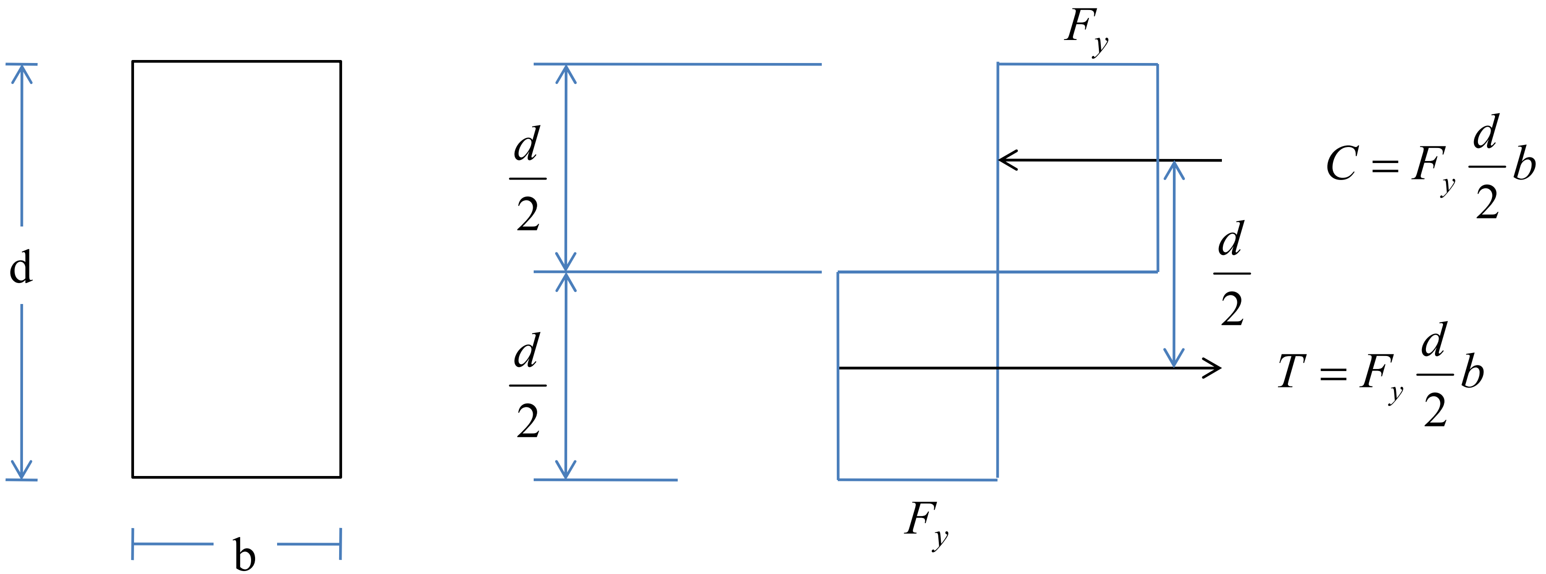
The Plastic moment M_p is equal to the yield stress F_y times the plastic section modulus S.

$$M_p = F_y \times S$$

The plastic section modulus equals $bd^2/4$ for a rectangular section.

$$M_p = F_y \frac{bd^2}{4}$$

The same value can be obtained by considering the resisting internal couple.



$$M_p = \left(F_y \frac{d}{2} b\right) \frac{d}{2} = \frac{F_y b d^2}{4}$$

The shape factor for this rectangular beam section is

$$\text{Shape Factor} = \upsilon = \frac{M_p}{M_y} = \frac{F_y S}{F_y Z} = \frac{S}{Z} = \frac{\frac{bd^2}{4}}{\frac{bd^2}{6}} = 1.5$$

At all stages of loading, the compression force C induced by the applied moment must equal the tension force T.

Or, the plastic neutral axis must be that axis which equally divides the area into two separate parts, i.e.

C = Compression Force (A_C x F_y)

T = Tension Force (A_T x F_y)

Force in Compression = Force in Tension

$$C = T$$

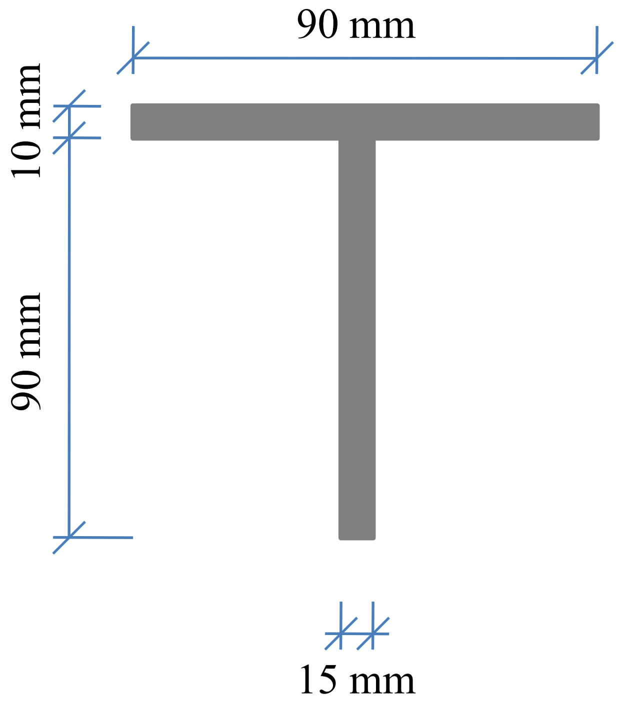
$$(A_C \times F_y) = (A_T \times F_y)$$

Area in compression = Area in Tension

In plastic analysis the neutral axis is the “equal area axis”.

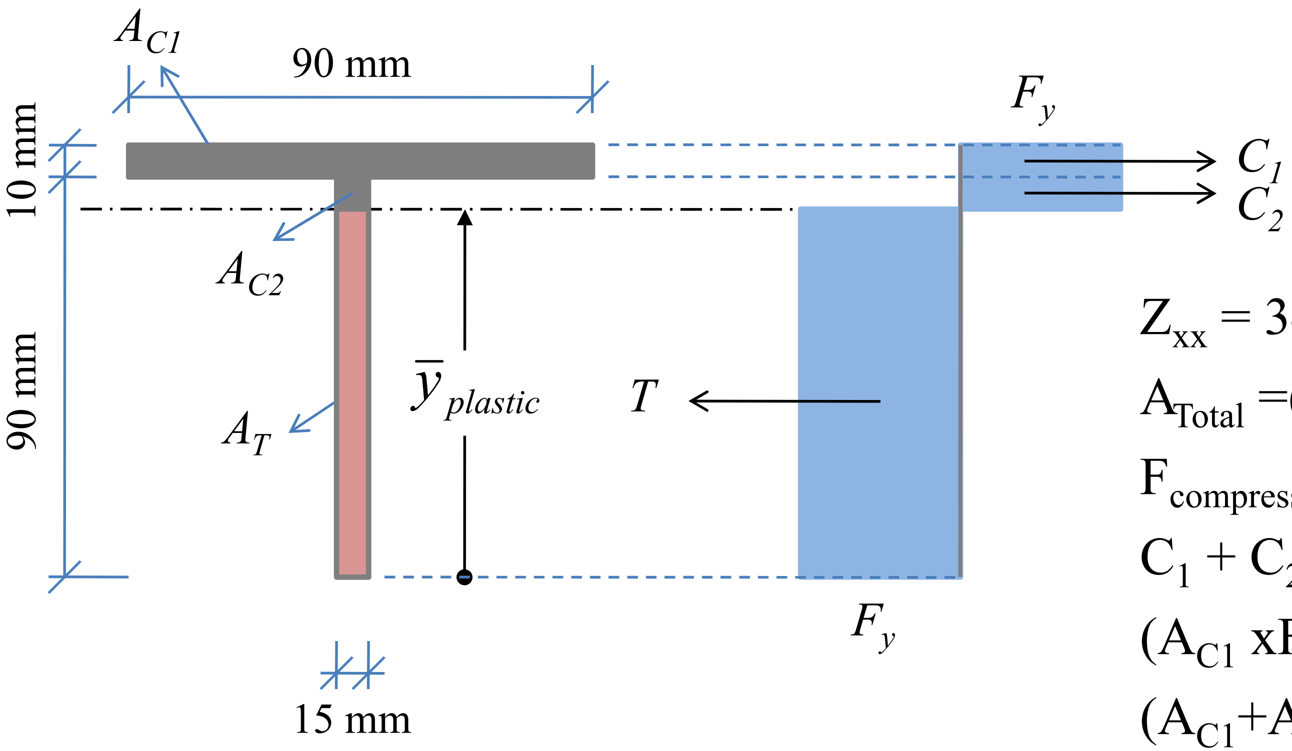
Example 1

Determine the position of the plastic neutral axis $\bar{y}_{plastic}$, the plastic section S_{xx}, and the shape factor υ for the welded section shown.



$Z_{xx} = 34.9 \times 10^3\ mm^3$

Solution:



$Z_{xx} = 34.9 \text{ x } 10^3 \text{ mm}^3$

$A_{Total} = (A_{C1} + A_{C2} + A_T)$

$F_{compression} = F_{tension}$

$C_1 + C_2 = T$

$(A_{C1} \text{ x} F_y) + (A_{C2} \text{ x} F_y) = (A_T \text{ x} F_y)$

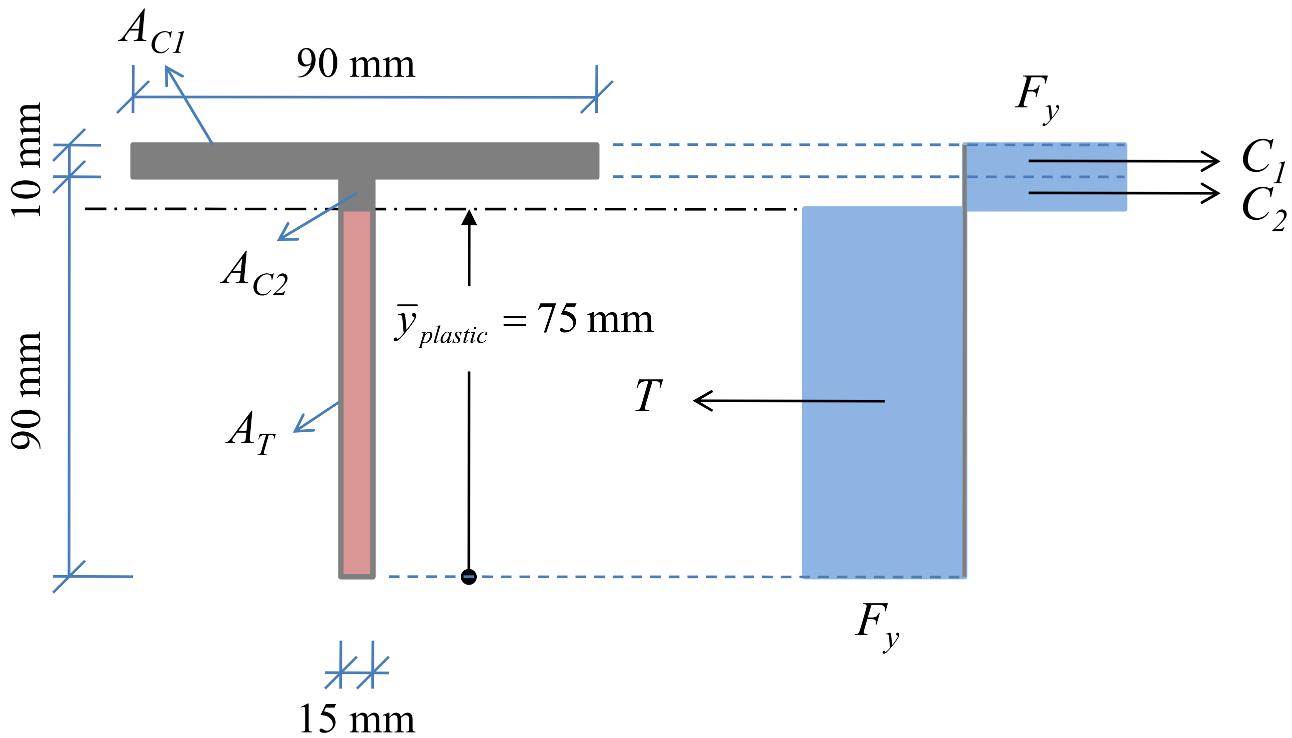
$(A_{C1} + A_{C2}) = A_T$

(1) Position of plastic neutral axis ($\bar{y}_{plastic}$)

$$A = [(90 \times 10) + (90 \times 15)] = 2250 \text{ mm}^2$$

$$\frac{A}{2} = \frac{2250}{2} = 1125 \text{ mm}^2$$

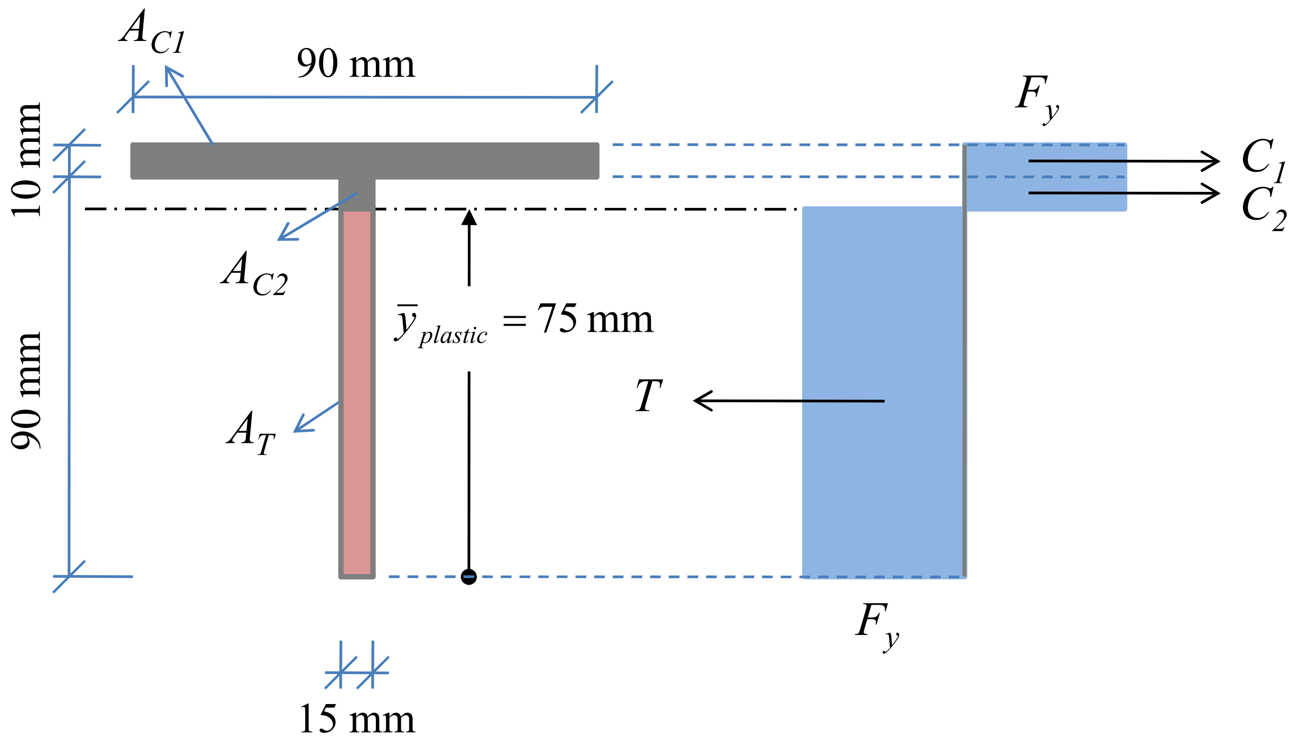
Solution:



For equal area axis:

$$\bar{y}_{plastic} = \frac{1125}{15} = 75 \text{ mm}$$

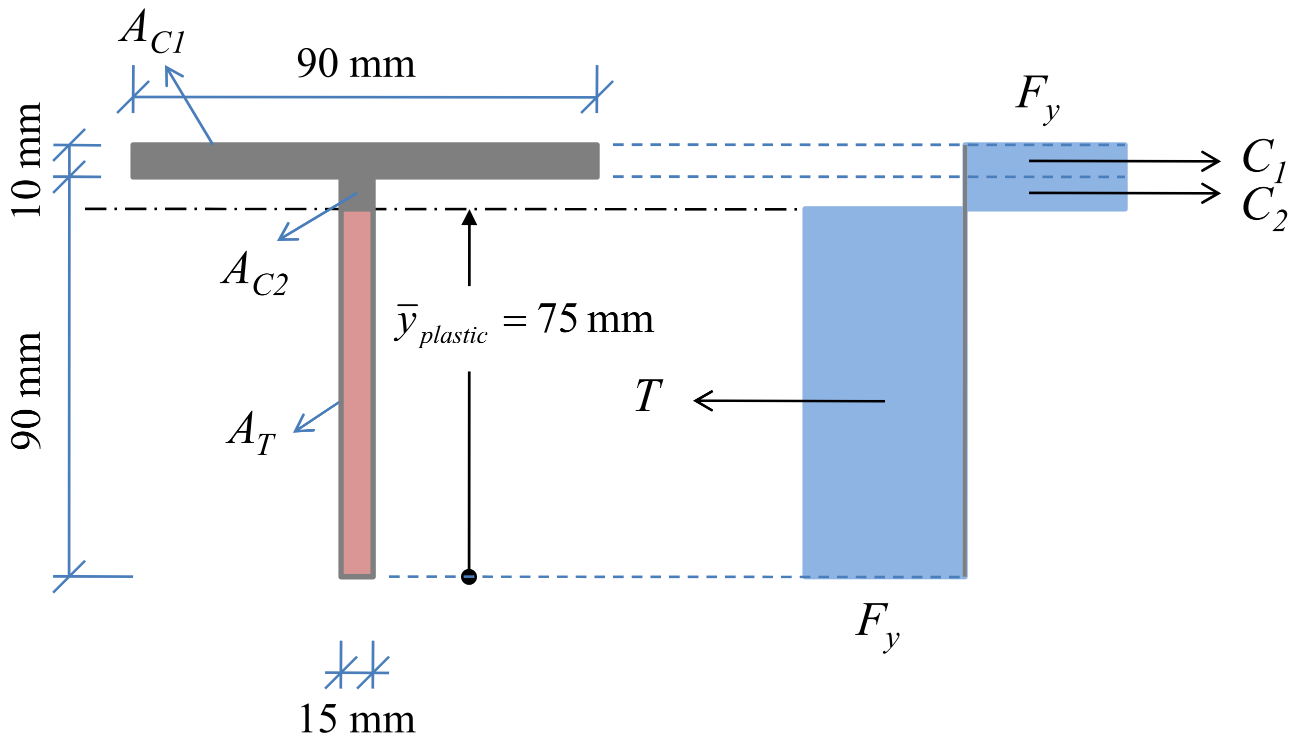
Solution:



(2) Plastic section modulus (S_{xx}): (first moment of area about plastic neutral axis)

$$S_{XX} = [(90\times 10)\times 20] + [(15\times 15)\times 7.5] + [(75\times 15)\times 37.5] = 61.875\times 10^3 \text{ mm}^3$$

Solution:



(3) Shape factor ($\upsilon = \frac{S_{xx}}{Z_{xx}}$)

$$\upsilon = \frac{S_{XX}}{Z_{XX}} = \left[\frac{61.875 \times 10^3}{34.9 \times 10^3}\right] = 1.77$$

The Load Factor

We wish to ensure that the load to which a structure is subjected is lower than the value which would cause collapse by a fairly large margin; we can define this margin using a load factor.

Suppose we have a structure subjected to a known working load "P". We can calculate that the onset of yielding would be caused by a load "P_y" and complete collapse by a load "P_c".

The load factor against collapse is:

$$\lambda = \frac{P_C}{P}$$

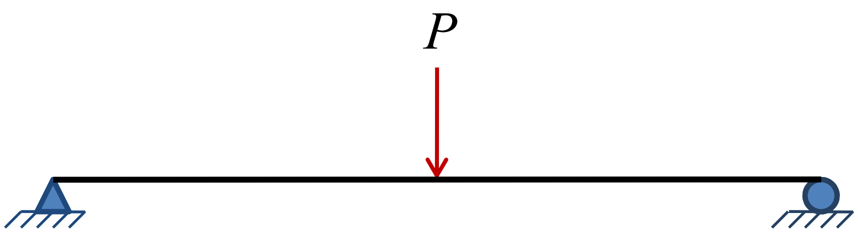
Safety factor against yielding is:

$$\gamma = \frac{P_y}{P}$$

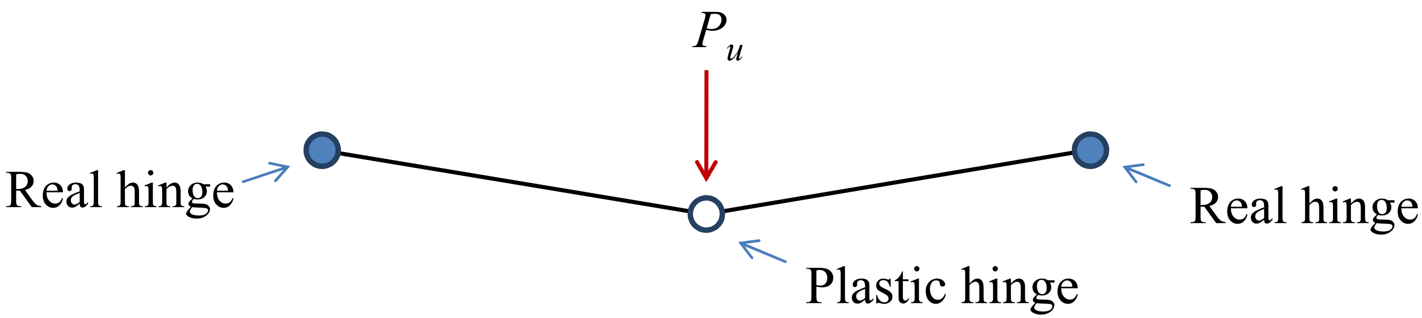
The Collapse Mechanism

A statically determinate beam will fail if one plastic hinge develops.

Consider the beam shown in figure.



Should the load be increased until a plastic hinge is developed at the point of maximum moment, an unstable structure will have been created.



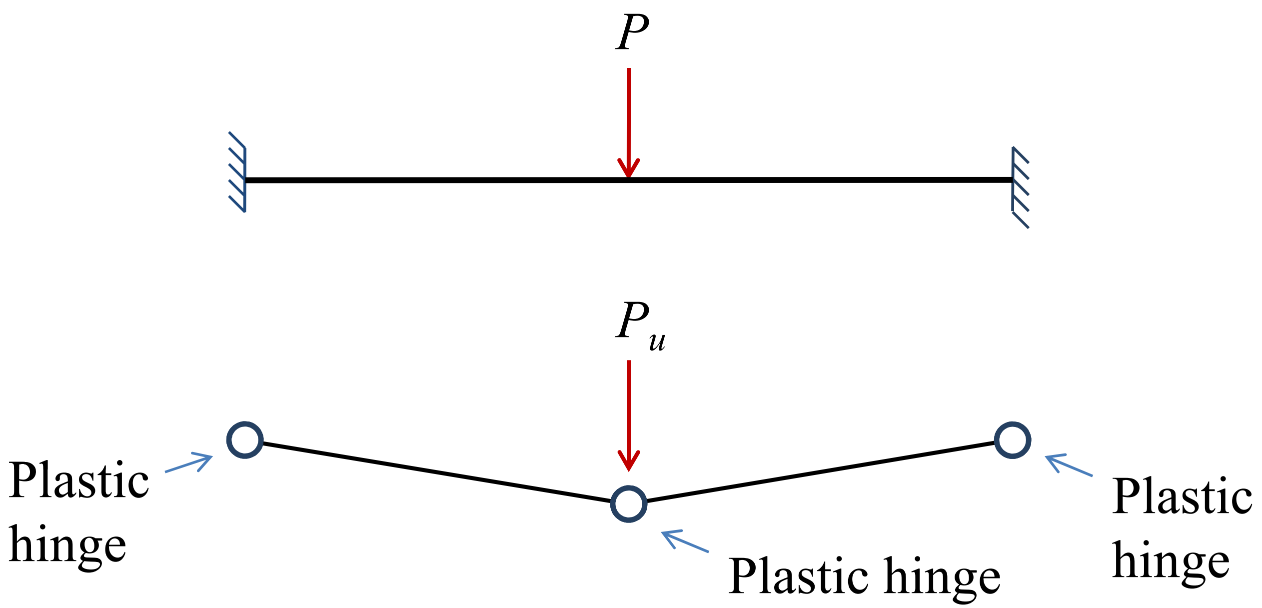
Any further increase in load will cause collapse.

The plastic theory is of little advantage for statically determinate beams and frames but it may be of decided advantage for statically indeterminate beams and frames.

For a statically indeterminate structure to fail it is necessary for more than one plastic hinge to form.

The number of plastic hinges required for failure of SIS vary from structure to structure, but may never be less than two.

The fixed-end beam of figure cannot fail unless the three plastic hinges are developed.



When a plastic hinge is formed in a SIS, the load can still be increased without causing failure if the geometry of the structure permits.

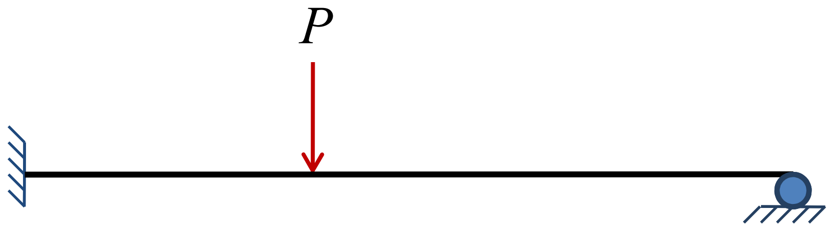
The plastic hinge will act like a real hinge insofar as loading increased is concerned.

As the load is increased there is a redistribution of moment because the plastic hinge can resist no more moment.

As more plastic hinges are formed in the structure, there will be eventually a sufficient number of them to cause collapse.

Actually some additional load can be carried after this time before collapse occurs as the stresses go into the strain hardening range, but the deflections that would occur are too large to be permissible.

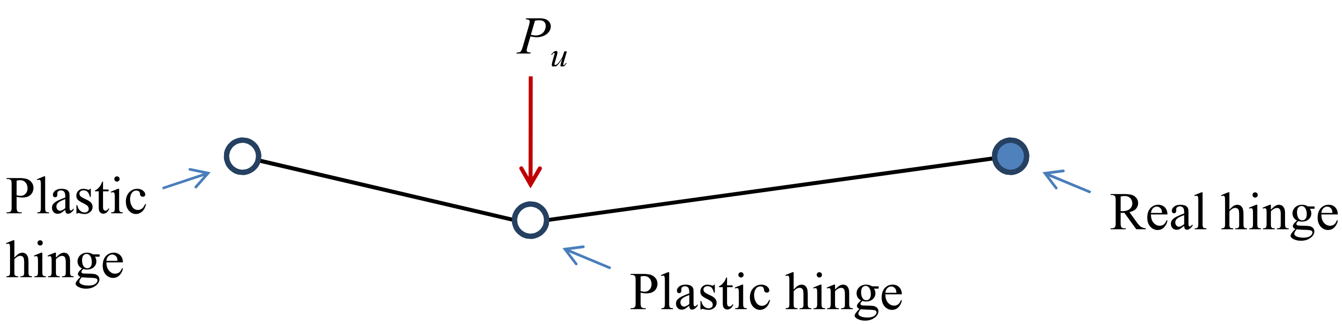
The propped beam of figure is an example of structure that will fail after two plastic hinges develop.



In this beam the largest elastic moment by the load is at the fixed end.

As the magnitude of the load is increased, a plastic hinge will form at that point.

The load may be further increased until the moment at some other point (underneath the point load) reaches the plastic moment.



Additional load will cause the beam to collapse.

The arrangement of plastic hinges and perhaps real hinges that permit collapse in a structure is called the mechanism.

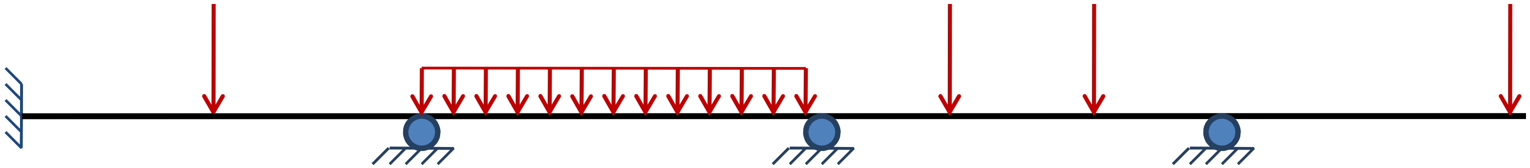
“When enough plastic hinges have formed in a structure to develop its full plastic load carrying capacity, then portions of the structure between hinges may displace without any further increase of load i.e., the portions between hinges behave as a mechanism.

Under these conditions the shape of the deformed body may be characterized as a straight line between any pair of hinges, known as Collapse mechanism.”

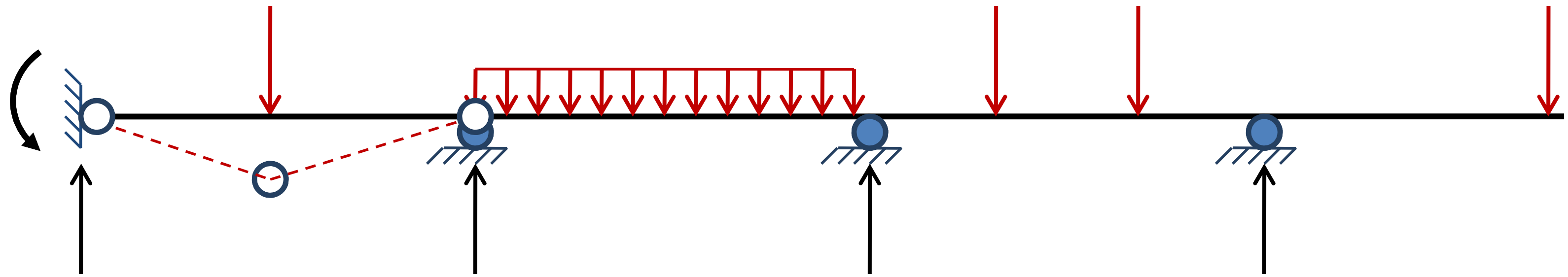
Partial Collapse

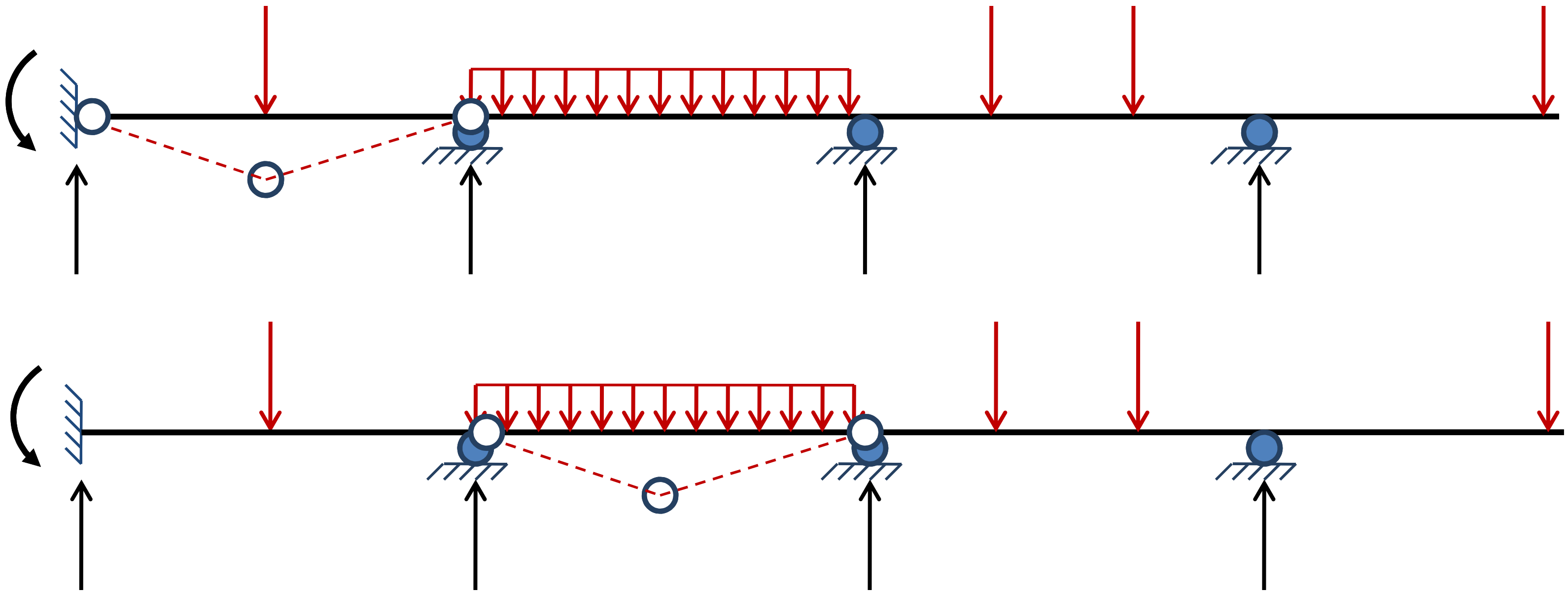
It is possible for part of a structure to collapse whilst the rest remaining stable.

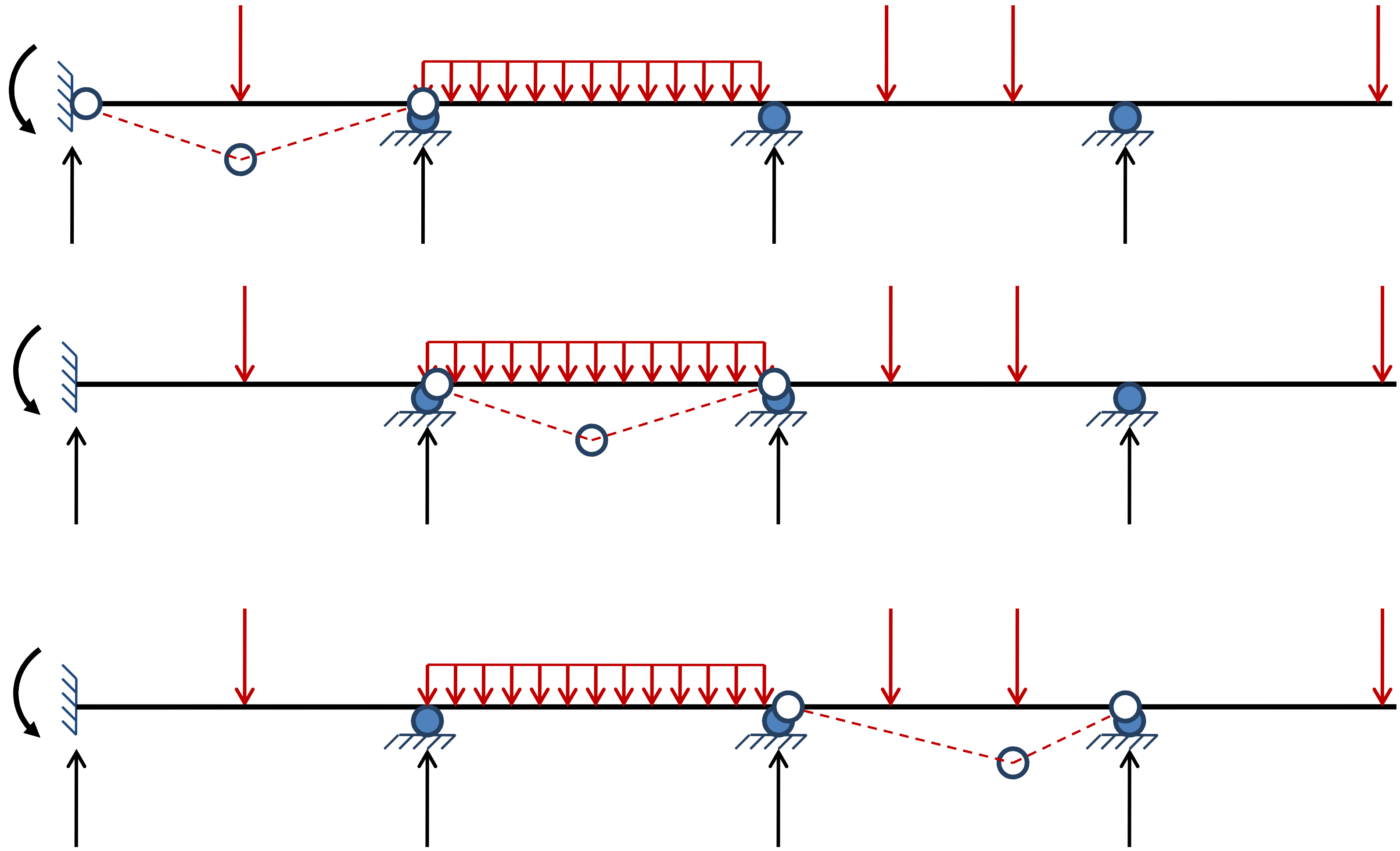
In this instance full collapse does not occur and the number of hinges required to cause partial collapse is less than (I_D+1).

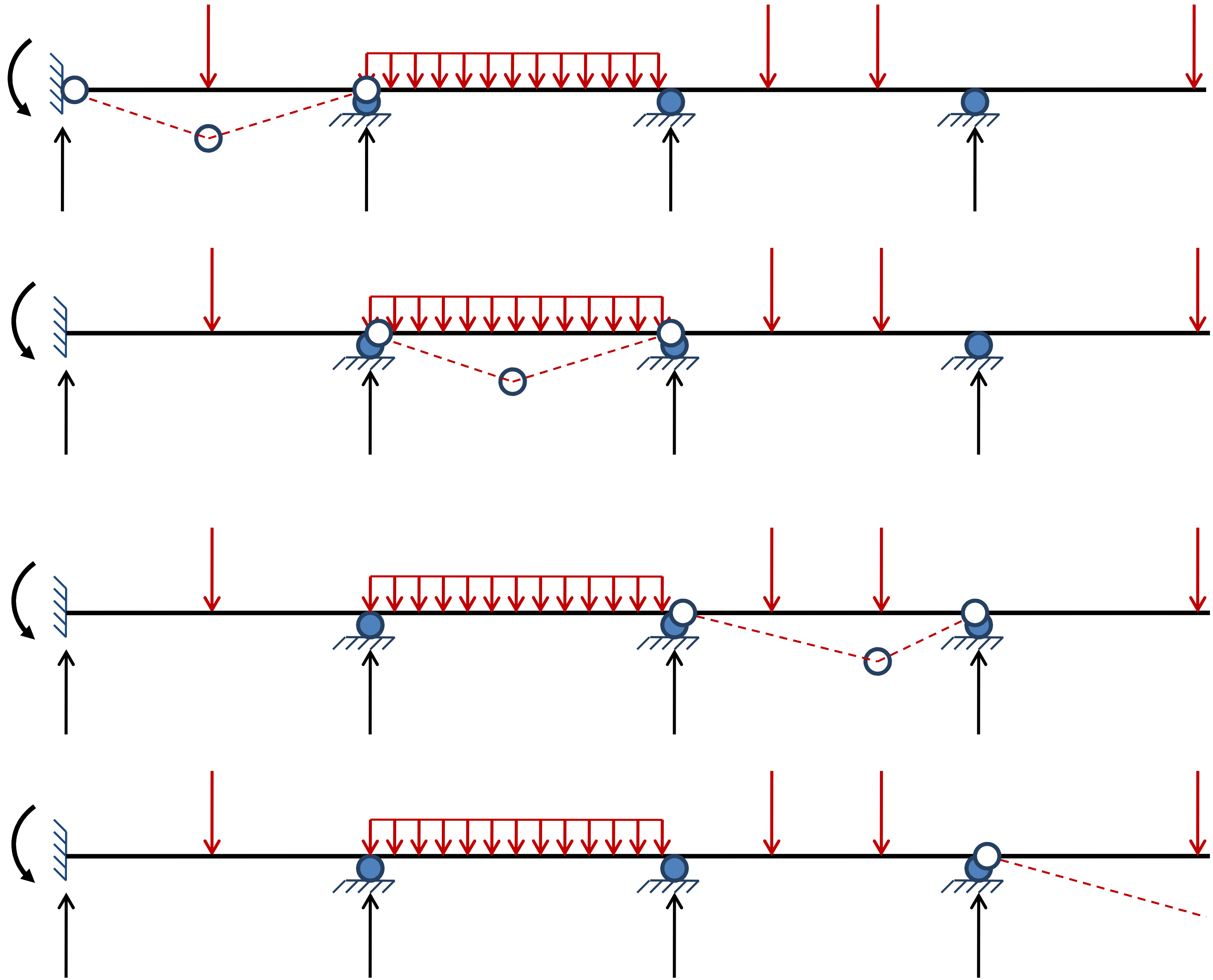


$$I_D = [(2m + r) - 2n] = [(2\times 4 + 5) - 2\times 5] = 3$$





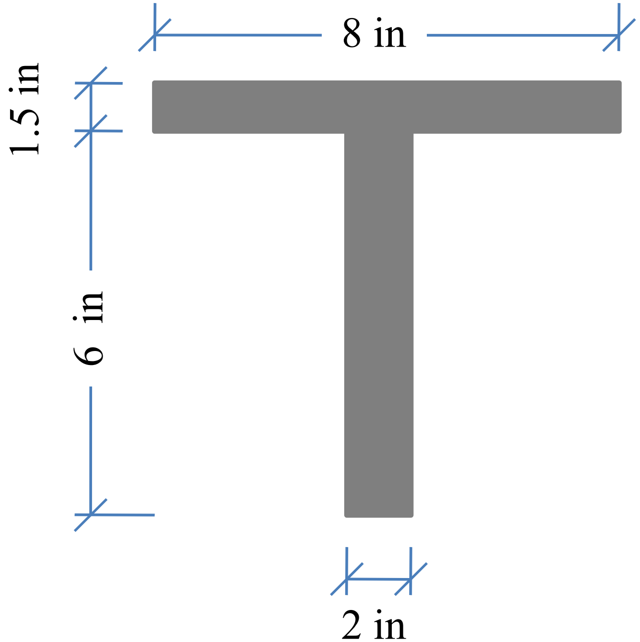


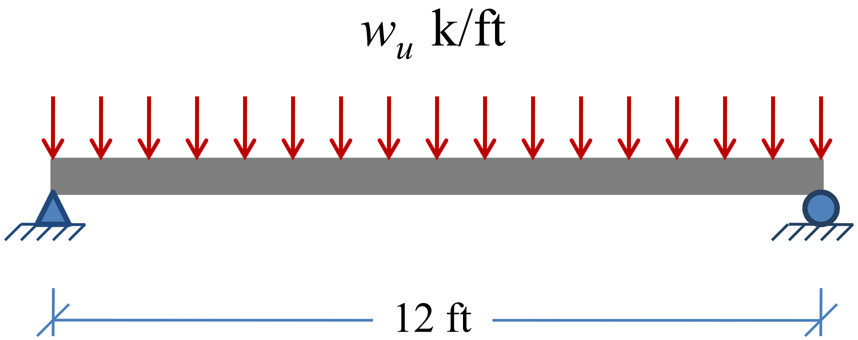


For any given design load applied to a redundant structure, more than one collapse mechanism may be possible.

Example 2

Determine M_y, M_p, and S for the steel tee beam shown. Also calculate the shape factor and the ultimate uniform load w_u that can be placed on the beam for a *12-ft* simple span. $F_y = 36$ ksi.





Solution:

Elastic Calculations:

$$A = (8)(1.5) + (6)(2) = 24 \text{ in}^2$$

$$\bar{y} = \frac{(12)(0.75) + (12)(4.5)}{24} = 2.625 \text{ in. from top flange}$$

$$I = \left(\tfrac{1}{3}\right)(2)\left(1.125^3 + 4.875^3\right) + \left(\tfrac{1}{12}\right)(8)\left(1.5^3\right) + (12)\left(1.875^2\right) = 122.4 \text{ in}^4$$

$$Z = \frac{I}{C} = \frac{122.4}{4.875} = 25.1 \text{ in}^3$$

$$M_y = F_y Z = \frac{(36)(25.1)}{12} = 75.3 \text{ k - ft} \qquad \text{ANS}$$

Solution:

Plastic Calculations:

Neutral axis is at the base of the flange.

$$S = (12)(0.75) + (12)(3) = 45 \text{ in}^3$$

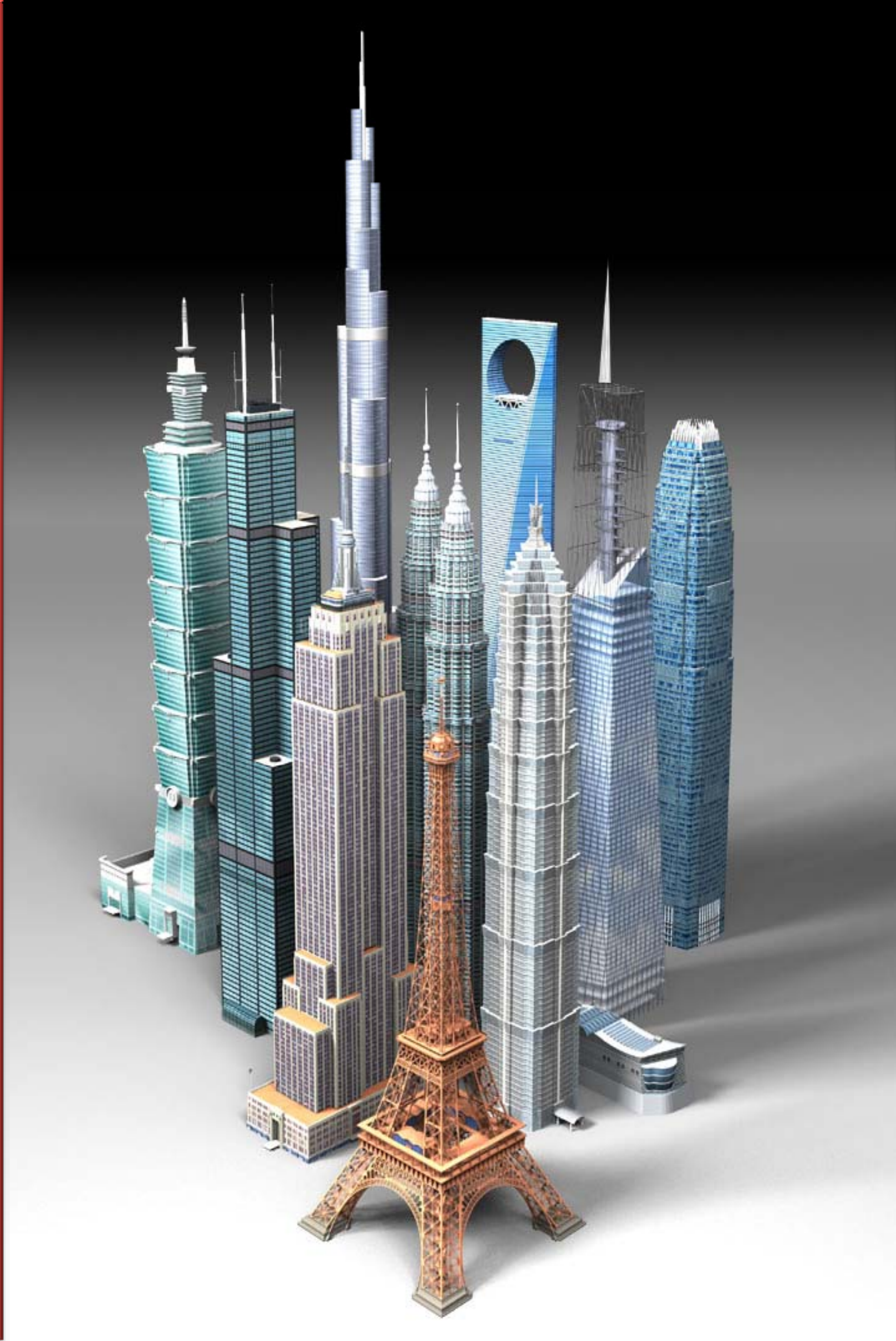
$$M_p = F_y S = \frac{(36)(45)}{12} = 135 \text{ k - ft}$$ ANS

$$\text{Shape Factor} = \frac{M_p}{M_y} \; or \; \frac{S}{Z} = \frac{45}{25.1} = 1.79$$ ANS

$$M_p = \frac{w_u L^2}{8}$$

$$w_u = \frac{(8)M_p}{L^2} = \frac{(8)(135)}{(12)^2} = 7.5 \text{ k/ft}$$ ANS

References:
-*Structural Analysis* by Jack. C. Mc.Cormac
-*Structures theory and analysis* by J. D. Todd
-*Examples in Structural Analysis* by W. M.C. McKenzie



End of Semester

Thank you